Nonasymptotic critical behavior from field theory at $d=3$. II. The ordered-phase case

C. Bagnuls
Service de Physique du Solide et de Résonance Magnétique, Centre d'Etudes Nucléaires de Saclay, 91191 Gif-sur-Yvette Cédex, France

C. Bervillier
Service de Physique Théorique, Centre d'Etudes Nucléaires de Saclay, 91191 Gif-sur-Yvette Cédex, France

D. I. Meiron
California Institute of Technology, Applied Mathematics, Pasadena, California 91125

B. G. Nickel
Department of Physics, University of Guelph, Guelph, Ontario, Canada N1G 2W1
(Received 27 March 1986)

We present the first detailed calculations in the ordered phase using the massive ϕ^4 field theory directly at $d=3$. It is shown that an adapted expansion allows the renormalization functions of the symmetric theory to be kept unchanged. Extending results in a previous paper [C. Bagnuls and C. Bervillier, Phys. Rev. B **32**, 7209 (1985)] (noted I), we obtain, for Ising-type systems, nonasymptotic functions of temperature for the spontaneous magnetization, the susceptibility, and the specific heat along the critical isochore, which include all the quantitative universal characteristics of critical behavior in the real preasymptotic critical domain D_{preas}. All universal leading- and first-correction amplitude combinations (including the new one $R_{B_{\mathrm{cr}}}^{-}$) are accurately estimated and are compared with previous theoretical and experimental estimates. We also show that the functions are well adapted to a suitable comparison with experiment and we describe how the adjustable parameters, limited to only three (the same as in I), enter in nonasymptotic critical behavior. Together with I, this work provides experimentalists with an efficient and coherent method which will facilitate the experimental test of the renormalization-group predictions.

I. INTRODUCTION

Universality in critical phenomena[1] is a concept which has allowed a very significant unification of the various second-order phase transitions. It has led to the classification of real systems according to the dimensionalities of both the space (d) and the order parameter (n). Each universality class is characterized by a unique (universal) set of critical exponents and of leading amplitude combinations. The renormalization-group (RG) theory, in particular in the field-theory framework at $d=3$,[2] has given very precise estimates for various of these universal quantities.[3–5] Experimentalists are interested in testing these quantitative predictions. However, difficulties (gravity, finite size, impurities, etc.) related to the approach to the critical point[6] make the comparison with theory a delicate matter, since the asymptotic critical domain, where pure power-law behavior holds, has a nonuniversal width. Of course, theory predicts nonanalytic corrections to scaling (Wegner expansion[7]) with universal characteristics (exponents and amplitude ratios), but, as in the case of the asymptotic pure power law, it does not yield any information on the convergence of this expansion. This disadvantage is accentuated by the presence of analytic corrections[8] for which theory could not provide information concerning their relative range of influence. Hence, experimentalists were obliged to make hypotheses, which were not always well controlled, concerning the corrections to scaling in order to reduce the number of adjustable parameters.[6] The challenge for theorists is to provide a theory involving the least number of free parameters as possible.

In Ref. 10, to which we shall refer as I, an accurate theoretical treatment of the ϕ^4 model in the field-theory framework at $d=3$ was presented. Nonasymptotic critical behaviors of the correlation length, the susceptibility, and the specific heat along the critical isochore above T_c were obtained. It was stressed that the validity of these results is *physically limited to the first nonanalytic correction to scaling* since those generated by higher transients (such as ϕ^6 and so on) were not taken into account. In particular, it is erroneous to pretend that the critical to classical crossover observed in the real systems can be determined by studying only the ϕ^4 Hamiltonian. However, it was shown that due to the accuracy of the work associated with the limited, but irreducible, number of adjustable parameters, it was possible to present a reliable method for measuring the size of the asymptotic and preasymptotic domains.[11,12] The theoretical functions so obtained, provided above T_c, a global presentation of universality of critical behaviors in D_{preas} (preasymptotic critical domain). To make the results of I more complete, it is necessary to extend the calculations to below T_c. The object of this paper is to present such calculations.

The starting point is the ϕ^4 bare Hamiltonian which reads

$$H\{\phi_0\} = \int d^dx \{ \tfrac{1}{2}[(\nabla\phi_0)^2(x) + r_0\phi_0^2(x)] + (g_0/4!)\phi_0^4(x) - h\phi_0(x)\} \ . \quad (1.1)$$

As recalled in I, we formally assume a cutoff Λ in the Feynman integrations. The presence of a magneticlike scaling field h is necessary to study the ordered phase, although we are interested in the critical isochore ($h=0$).

In Ref. 13, we sketched the manner in which the massive field-theory framework could be extended to the ordered-phase case. We showed that it was possible, using suitable expansion and renormalization, to study this case using the same long series[14] as those used in I. We present here the details and justifications of these calculations and extend them up to fifth-loop order for Ising-type systems ($n=1$). Due to symmetry breaking, the study of the O(n) symmetry case, with n greater than one, is complicated at the calculational level (Feynman integrals with two different masses[15]) although the renormalization scheme should be the same as that above T_c.[16] It also seems interesting to consider the case $n=2$, which corresponds to the superfluid phase transition of ^{4}He,[17] which is well adapted to the study of the critical singularities. Although they are not impossible to carry out, the calculations for $n>1$ are, below T_c, much more tedious than are those for the case $n=1$. Indeed, we think that it will be possible to make measurements in liquid-gas systems closer to the critical point by decreasing the strength of the gravitational field (experiments made in space). Hence Ising-type systems will generate almost the same interest as ^{4}He.

From the knowledge of one order more than that in Ref. 13, we obtain a better accuracy for the universal combinations of leading amplitudes. We can perform a nonasymptotic study similar to that made in I. We estimate the universal ratios of the first correction amplitudes at $d=3$ and the universal quantity $R_{B_{\mathrm{cr}}}^-$.[18,19] All these universal characteristics are quantitatively contained, as in I, in the nonasymptotic functions which are proposed to describe the critical behavior of the spontaneous magnetization M_0^s, of the susceptibility χ^- and of the specific heat C^- along the critical isochore below T_c (see Sec. V). The correlation length below T_c has not been considered here. This would require calculations of supplementary integrals which have not been calculated.

This is the first time that calculations in the ordered phase are made with accuracy using the massive ϕ^4 field theory directly at $d=3$. The previous knowledge[20,21] on this phase was obtained in the ϵ-expansion framework up to two or three loops and had limited accuracy (see Sec. V and Appendix C).

The organization of the paper is as follows. In Sec. II, we recall the main steps of I. Particular attention is paid to the definition of a linear temperaturelike scaling field (t) in the massive theory. It is explicitly shown how the correlation functions of the ordered phase are to be expanded in order to keep unchanged the definitions of the renormalization functions Z_i introduced for the symmetric theory ($T>T_c$). The two scaling fields t_+ and t_- relative to the two phases are expressed as functions of the *same renormalized coupling constant g*. They are shown to be proportional to each other at the fixed point value g^* of g. In Sec. III, we give the perturbative expressions, up to fifth-loop order, of M_0^s, χ^-, and C^-. A particular renormalization at $d=3$, unrelated to critical behavior, is explicitly introduced to get the perturbative expansion of the singular part of the free energy which also is explicitly given. The values at $d=3$ of the Feynman integrals corresponding to the one-vertex irreducible graphs of the free energy are given in Table I. After introducing the renormalization relevant to the study of critical phenomena (renormalization of $d=4$), it is shown that the critical singularities of M_0^s, χ^-, and C^- may be factorized out through combinations of the renormalization functions $Z_i(g)$ ($i=1-3$). The universal combinations of leading critical amplitudes are expressed as combinations of series in powers of g which are nonsingular at g^*. In Sec. IV, after noticing some simplification occurring at g^*, the series are resummed at $g=g^*$ to get estimates, at $d=3$, of the leading critical amplitude combinations which are displayed in Table VI. In Sec. V, we present the nonasymptotic treatment of M_0^s, χ^-, and C^- at values of g different from g^*. We then determine the nonasymptotic functions of t_- for these three measurable quantities in a form similar to that of I. The universal ratios of first-confluent correction amplitudes are determined and presented in Table VIII. Comparison with experiments and with ϵ expansion is made. In Sec. VI, we describe the way the theoretical functions, obtained in the preceding section, are to be compared with experimental data. It is shown that the adjustable parameters introduced in I (only three for the complete set of measurable quantities above and below T_c) correspond to the usual process of reducing the measured quantities to dimensionless[23] quantities. Particular attention is paid to the specific-heat case. First we discuss the definition of the "regular part" given in I. Then we stress again the limited validity of the ϕ^4 theory. In particular, we recall the possible presence of large analytic corrections, which are usually neglected[6] in the experimental data analysis (see Ref. 12). The conclusion is given in Sec. VII. Additional technical details are given in the Appendices.

The reader who should be particularly interested in the comparison of the results of the present work with experiments may directly refer to Secs. V and VI without considering the technical details of the calculations presented in the preceding sections. He will find the definition of the theoretical functions which reproduce the evolution in temperature (below T_c) of M_0^s, χ^-, and C^- in the ϕ^4 theory [Eq. (5.5) and Table VII]. The use of these functions in a comparison with experiments is described in detail in Sec. VI. The estimates of the universal amplitude combinations are displayed in Tables VI, VIII, and IX.

II. LINEAR SCALING FIELD AND RENORMALIZATION

In this section, we show that it is possible to choose a renormalization scheme below T_c which does not modify the renormalization functions known up to sixth loop or-

TABLE I. Numerical values, directly at $d=3$ and up to fifth-loop order, of one-vertex irreducible Feynman integrals associated to the graphs drawn by Heap in Ref. 22. The graphs are classified with respect to the loop number b. In Ref. 22, graphs with $b=2,3$ are given in Table III, $b=4$ in Table IV and $b=5$ in Table V. The number h corresponds to the graph number attributed in the paper of Heap; $2l$ is the number of three-leg vertices and P_m, the symmetry factor relative to each graph. We have numbered the graphs corresponding to the same b and same l by m in the third column. The value displayed in the sixth column corresponds to $I_m^{(b,l)}(\epsilon=1)$ after 3-d renormalization (see Sec. III and Appendix A). Let us note that y stands for r_0'/g_0^2. As an example, a graph of Fig. 1(a) is characterized in this table by $b=4$, $h=3$ (Table IV of Heap), and $l=0$. It is the only graph with $h=3$ and $l=0$, which implies $m=1$. Its symmetry factor is 48. The case of graphs ($b=5$, $l=2$, and $m=1$, or 2) is explicitly considered in Appendix A.

b	h	l	m	P_m^{-1}	Value at $d=3$
1	—	0	1	2	$-4/3$
2	1	1	1	12	$-2\ln(y)$
3	1	0	1	48	$-22.794\,173\,68 + 16\ln(y)$
	2	1	1	8	4.107 471 254
	3	2	1	16	0.519 431 241 3
	4		2	24	0.173 900 610 70
4	3	0	1	48	$-19.739\,208\,80\ln(y)$
	5	1	1	24	$-0.296\,452\,724\,0 - (4/3)\ln(y)$
	6		2	16	2.065 719 357 1
	7		3	8	1.723 490 549 7
	8		4	8	1.240 596 097 8
	9	2	1	16	0.430 513 113 60
	10		2	4	0.311 603 130 40
	11		3	4	0.125 786 539 70
	12		4	8	$0.795\,169\,089\,00\times 10^{-1}$
	13	3	1	48	0.183 616 246 10
	14		2	16	$0.807\,212\,429\,00\times 10^{-1}$
	15		3	8	$0.378\,597\,287\,00\times 10^{-1}$
	16		4	12	$0.146\,202\,458\,00\times 10^{-1}$
	17		5	72	$0.122\,446\,700\,00\times 10^{-1}$
5	15	0	1	144	$4.218\,251\,52 + 4.602\,913\,152\ln(y) + 4\ln^2(y)$
	16		2	128	22.909 318 39
	17		3	32	16.602 295 22
	31	1	1	48	$-0.142\,393\,552 - 1.038\,862\,482\ln(y)$
	32		2	12	$-0.122\,911\,141 - 0.767\,152\,192\ln(y)$
	33		3	32	1.373 092 004
	34		4	16	2.449 689 513
	35		5	8	1.072 299 357
	36		6	16	0.629 580 783
	37		7	8	0.785 125 191
	38		8	4	0.729 050 922
	39		9	8	0.853 563 223
	40		10	4	0.489 725 124
	41		11	12	0.318 157 28
	49	2	1	48	See Eq. (A13)
	50		2	24	
	51		3	24	$-0.007\,497\,124 - 0.086\,950\,305\ln(y)$
	52		4	32	0.369 272 786 80
	53		5	64	0.369 272 786 80
5	54	2	6	8	0.226 029 376 10
	55		7	8	0.263 941 873 70
	56		8	8	$0.979\,101\,693\,00\times 10^{-1}$
	57		9	32	0.290 975 627 80
	58		10	8	0.263 941 873 70
	59		11	16	0.226 029 376 10
	60		12	16	0.222 945 449 60
	61		13	4	0.164 047 265 30
	62		14	16	0.196 217 896 90
	63		15	8	$0.948\,950\,873\,00\times 10^{-1}$
	64		16	8	0.189 568 128 60
	65		17	8	$0.934\,864\,606\,00\times 10^{-1}$
	66		18	4	$0.948\,950\,873\,00\times 10^{-1}$
	67		19	16	$0.979\,101\,693\,00\times 10^{-1}$
	68		20	8	0.189 568 128 60
	69		21	4	$0.665\,917\,600\,00\times 10^{-1}$
	70		22	8	$0.532\,161\,929\,00\times 10^{-1}$
	71		23	16	$0.499\,475\,400\,00\times 10^{-1}$
	72		24	4	0.171 562 451 10
	73		25	4	$0.727\,466\,958\,00\times 10^{-1}$
	74		26	2	$0.591\,611\,841\,00\times 10^{-1}$
	75		27	4	$0.565\,051\,900\,00\times 10^{-1}$
	76		28	8	$0.895\,561\,236\,00\times 10^{-1}$
	77		29	12	$0.329\,509\,400\,00\times 10^{-1}$
	78		30	2	$0.348\,176\,300\,00\times 10^{-1}$
	79		31	16	$0.343\,553\,904\,00\times 10^{-1}$
	80	3	1	32	0.169 301 035 39
	81		2	16	$0.707\,502\,835\,80\times 10^{-1}$
	82		3	16	$0.335\,564\,678\,10\times 10^{-1}$
	83		4	16	$0.672\,450\,000\,60\times 10^{-1}$
	84		5	8	$0.707\,502\,835\,80\times 10^{-1}$
	85		6	8	0.117 376 605 85
	86		7	4	$0.305\,215\,948\,00\times 10^{-1}$
	87		8	8	$0.335\,564\,678\,10\times 10^{-1}$
	88		9	4	$0.501\,331\,498\,30\times 10^{-1}$
	89		10	8	$0.293\,503\,973\,20\times 10^{-1}$
	90		11	4	$0.305\,215\,948\,00\times 10^{-1}$
	91		12	4	$0.236\,749\,072\,00\times 10^{-1}$
	92		13	4	$0.121\,271\,809\,70\times 10^{-1}$
	93		14	2	$0.116\,195\,071\,50\times 10^{-1}$
	94		15	8	$0.991\,101\,177\,00\times 10^{-2}$
	95		16	8	$0.389\,670\,857\,60\times 10^{-1}$
	96		17	4	$0.174\,204\,453\,00\times 10^{-1}$
	97		18	4	$0.160\,620\,445\,50\times 10^{-1}$
	98		19	4	$0.185\,067\,728\,40\times 10^{-1}$
	99		20	2	$0.683\,283\,800\,00\times 10^{-2}$
	100		21	8	$0.794\,861\,819\,00\times 10^{-2}$
	101		22	4	$0.579\,484\,517\,00\times 10^{-2}$
	102		23	4	$0.541\,938\,733\,00\times 10^{-2}$
	103	4	1	128	$0.992\,567\,553\,97\times 10^{-1}$
	104		2	16	$0.344\,407\,886\,78\times 10^{-1}$
	105		3	96	$0.271\,438\,313\,55\times 10^{-1}$

TABLE I. (*Continued*).

b	h	l	m	P_m^{-1}	Value at $d=3$	b	h	l	m	P_m^{-1}	Value at $d=3$
5	106	4	4	16	$0.167\,658\,921\,10\times10^{-1}$	5	113	4	11	16	$0.365\,154\,744\,03\times10^{-2}$
	107		5	8	$0.114\,471\,496\,11\times10^{-1}$		114		12	16	$0.315\,887\,574\,11\times10^{-2}$
	108		6	16	$0.137\,765\,510\,25\times10^{-1}$		115		13	4	$0.165\,636\,667\,13\times10^{-2}$
	109		7	32	$0.100\,335\,687\,85\times10^{-1}$		116		14	12	$0.139\,706\,003\,40\times10^{-2}$
	110		8	8	$0.658\,470\,742\,72\times10^{-2}$		117		15	48	$0.121\,514\,553\,40\times10^{-2}$
	111		9	4	$0.415\,352\,701\,57\times10^{-2}$		118		16	16	$0.117\,156\,814\,90\times10^{-2}$
	112		10	8	$0.478\,648\,271\,39\times10^{-2}$						

der.[14] In particular, we give the nonasymptotic expression of the linear temperaturelike scaling field t_- in terms of g.

A. General considerations

In I, the case above T_c was treated in detail. It was shown that it is possible to neglect the cutoff dependence of the bare theory provided that renormalization, which removes the logarithmic ultraviolet (uv) divergences[16] occurring at $d=4$ is carried out at $d=3$. The limit $\Lambda\to\infty$ at $d=3$ in the bare theory leads, however, to uv divergences which necessitate an extrarenormalization not relevant to the study of critical behavior. A coherent theoretical scheme for obtaining the critical behavior above T_c was defined according to the following steps:

(i) In order to avoid uv singularities occurring at $d=4-2/k$ (Ref. 24) ($k=1,2,3,\ldots$), we make an analytic continuation in d. We may then drop Λ in the bare theory.

(ii) We eliminate the poles at $\epsilon=1$ by means of an extrarenormalization at $d=3$. The finite bare theory at infinite Λ and at $d=3$ so obtained is considered as the physical theory (i.e., the bare quantities are dimensioned by powers of Λ).

(iii) We introduce a linear scaling field (definition of T_c).

(iv) The renormalization transformations which eliminate the logarithmic uv divergences at $d=4$ are introduced.

(v) After obtaining the critical behavior for the renormalized theory we return to the bare theory by the inverse renormalization transformations.

(vi) The cutoff dependence is reintroduced through adjustable parameters so as to obtain the realistic bare correlation functions.

We stress the difference that exists between the renormalization of $d=4$ and that of $d=3$. The first renormalization concerns the elimination of logarithmic (at $d=4$) uv divergences through renormalizations of the coupling constant (g_0), scaling field (t), and fluctuating order parameter (ϕ_0) which, respectively, introduce the renormalization functions Z_1, Z_2, and Z_3 [see Eqs. (2.1) and (2.2) below]. Indeed, these uv divergences are, for dimensional reasons, tied to infrared (ir) divergences in which we are interested for the study of critical phenomena. Although set up to render the ϕ^4 theory free of uv divergences at $d=4$, the renormalization functions Z_i are calculated at $d=3$.[2] In the following, we shall refer to this renormalization scheme as the 4-d renormalization.

The origin of the renormalization of $d=3$ (referred to, in the following, as the 3-d renormalization) is very different. It stems from the definition of the bare theory at infinite cutoff and is justified only after having studied the finite cutoff effects in the critical behavior (see I). The arbitrariness related to 3-d renormalization is inherent to the RG approach which gives a description of real systems only via universality classes. In a given universality class, different real systems will have, in addition to different critical amplitudes, different critical temperatures (T_c) and different regular behaviors (e.g., the regular part $\mathscr{F}_{\rm reg}$ of the free energy). The elimination of the poles at $d=3$ of above step (ii) is performed in Sec. III and Appendix A. It corresponds to the definition of a linear scaling field (definition of T_c) and, *as far as only critical fluctuations are concerned,* to a zero regular part for the free energy ($\mathscr{F}_{\rm reg}=0$).[25] We stress the fact that the definition of T_c and a choice for $\mathscr{F}_{\rm reg}$ eliminate the arbitrariness introduced by 3-d renormalization without modifying the critical behavior.

In summary, the two renormalization schemes may be related to two sets of adjustable parameters (see Sec. VI):

(i) 3-d renormalization is, in principle, associated with noncritical adjustable parameters: namely the definition of T_c and of $\mathscr{F}_{\rm reg}$. By noncritical adjustable parameters we mean quantities which could be experimentally determined independently from the critical behavior itself. We will see (Sec. VI) that, for comparison with experiments, $\mathscr{F}_{\rm reg}$ is not equal to zero due to the degrees of freedom uncoupled to the critical fluctuations. This will induce a noncritical regular part $B_{\rm bg}$ in the specific heat which should be determined, at least approximately, from experiments far from T_c.[25]

(ii) 4-d renormalization is associated to the nonuniversal scales of the temperaturelike and magneticlike scaling fields and to a length scale. This involves the complete set of physical variables needed to carry out Wilson's renormalization process[8] which relates a change of the length scale to that of the scales of two scaling fields.

It has been recalled in I that the use of a linear scaling field t in the renormalization process introduces spurious infrared (ir) divergences in the perturbative expansion for $d<4$. This is cured in the massive field theory where the critical point approach is described by the correlation length (a nonlinear scaling field) ξ. This approach is indeed controlled by the way the renormalized coupling constant g goes to the fixed point g^* ($\xi\to\infty$). Critical behavior of quantities of interest were obtained as implicit functions of t_+ through their g dependence and the restoration of $t_+(g)$ (a nonperturbative quantity) from the resummation of the perturbative expansions.

The main characteristic of RG theory, which we shall use in the following, is that one gets a finite spontaneous symmetry-breaking theory ($T<T_c$) from having simply renormalized the symmetric theory ($T>T_c$).[15,16] This property is responsible for the reduction of the complete set of critical exponents (above and below T_c) to only two independent critical exponents. These are related to the singularities, at the fixed point g^*, of Z_2 and Z_3. As for Z_1, it serves to define g^* itself.[16] In going below T_c, the renormalization scheme will be essentially unchanged. Indeed, no new uv divergence (neither at $d=4$ nor at $d=3$) is introduced by the spontaneous symmetry breaking effect. Only the expressions of the correlation functions are modified by the presence of a nonzero magnetization. This is conveyed, at the calculational level for $n=1$, by the presence of three-leg vertices in addition to the usual four-leg vertices. Before considering the ordered phase case, it is thus useful to reproduce the main equations obtained in I.

B. Disordered phase case ($T>T_c$)

Let us recall briefly the main steps of calculations given in I with particular attention to the expression of the linear scaling field t_+ as function of the renormalized coupling constant g. In the following, we shall suppose that 3-d renormalization has already been performed. Hence the bare correlation functions will no longer depend on the cutoff Λ and the limit $d\to 3$ can be safely used. In practice, this is realized by shifting the bare mass r_0 to r_0' by the amount $\delta r_0(\epsilon)$ which subtracts the poles at $\epsilon=1$ (see I and Appendix A).

The renormalization transformations relevant to critical phenomena (4-d renormalization) are then introduced:

$$\phi_0(x)=[Z_3(g)]^{1/2}\phi_R(x)\ , \tag{2.1a}$$

$$g_0=mgZ_1(g)/[Z_3(g)]^2\ , \tag{2.1b}$$

$$[\phi_0(x)]^2=[Z_3(g)/Z_2(g)][\phi_R(x)]^2_R\ , \tag{2.1c}$$

with,

$$[Z_3(g)]^{-1}=\frac{\partial}{\partial p^2}\Gamma^{(0,2)}_{0,+}(\{p\};r_0',g_0)\,|_{p^2=0}\ , \tag{2.2a}$$

$$[Z_1(g)]^{-1}=\Gamma^{(0,4)}_{0,+}(\{0\};r_0',g_0)/g_0\ , \tag{2.2b}$$

$$[Z_2(g)]^{-1}=\Gamma^{(1,2)}_{0,+}(\{0\};r_0',g_0)\ . \tag{2.2c}$$

The subscripts 0, + refer to the bare correlation functions above T_c. To be complete, we also give the definition of m which reads

$$r_0'=m^2/Z_3(g)+\delta\overline{m}^{\,2} \tag{2.3}$$

with $\delta\overline{m}^{\,2}$ chosen such that

$$\Gamma^{(0,2)}_{0,+}(\{0\};r_0',g_0)=m^2/Z_3(g)\ . \tag{2.4}$$

It is worthwhile to make some comments on the meaning of the Eqs. (2.1)—(2.4). In principle, we have replaced the bare variables r_0',g_0 by m and g [Eqs. (2.3) and (2.1b)]. However, from Eq. (2.1b), m may, as well, be considered as a function of g and g_0. It is also the case for r_0' through Eqs. (2.3) and (2.4). In the following we shall adopt this way of considering the above transformations. Let us notice that g_0 thus remains the only dimensioned parameter. This explains why the dimensionless renormalization functions Z_i ($i=1$, 2, and 3) depend only on g in Eqs. (2.2).

The quantities directly accessible in perturbation theory (powers series in g) are the bare correlation functions. In particular, the functions $Z_i(g)$ [Eqs. (2.2)] have been calculated up to sixth-loop order by Nickel *et al.*[14] This provides the opportunity, using Eq. (2.1b), to quantitatively describe the way that m goes to zero when $g\to g^*$ (see I). We now try to determine how the linear scaling field $t=r_0'-r_{0c}'$ vanishes with $g-g^*$. In order to make the answer for the ordered-phase case as clear as possible, let us again derive, in detail, the expression of $t_+(g)$ obtained in I.

From the definition of the mass insertion,[16] Eq. (2.2c) may also be written as follows:

$$[Z_2(g)]^{-1}=\frac{\partial}{\partial r_0'}\Gamma^{(0,2)}_{0,+}(\{0\};r_0',g_0)\,|_{g_0}\ . \tag{2.5}$$

By using Eq. (2.4), in which m will be considered as a function of g and g_0 through Eq. (2.1b), we obtain

$$[Z_2(g)]^{-1}=g_0^2\left.\frac{\partial g}{\partial r_0'}\right|_{g_0}\frac{d}{dg}\left[\frac{[Z_3(g)]^3}{[gZ_1(g)]^2}\right]\ . \tag{2.6}$$

Since $r_0'(g)$ is also dimensioned by g_0, this equation may be rewritten,

$$\frac{d(r_0'/g_0^2)}{dg}=Z_2(g)\frac{d}{dg}\left[\frac{[Z_3(g)]^3}{[gZ_1(g)]^2}\right]\ . \tag{2.7}$$

Finally, by using the condition that the dimensionless scaling field $t_+^*=(r_0'-r_{0c}')/g_0^2$ vanishes at the fixed point g^*, we obtain the desired result,

$$t_+^*(g)=-\int_g^{g^*}dx\,Z_2(x)\frac{d}{dx}\left[\frac{[Z_3(x)]^3}{[xZ_1(x)]^2}\right]\ . \tag{2.8}$$

This expression, as explained in I, has no Taylor expansion around $g=0$. This justifies the use of the massive framework of the renormalization instead of the μ renormalization[26] framework. The latter, which maintains the linearity of the temperaturelike scaling field t_+ within the perturbative expansion, leads to spurious ir divergences. Of course this process, which implies ϵ expansion, allows calculation of the critical exponents. Le Guillou and Zinn-Justin have recently shown[27] that the estimates for the critical exponents obtained from the massive field theory directly at $d=3$ and from series which are long enough in powers of ϵ (Ref. 28) are in agreement. However the incompatibility, noted above, between perturbation and linearity of the temperaturelike scaling field introduces spurious singularities at the rationals $\epsilon=2/k$. Hence ϵ is limited to infinitesimal values.[2] Let us note that this problem is different from that of the divergence of the series which implies use of resummation methods to get numbers, at $\epsilon=1$, from expansion around $\epsilon=0$. Of course the spurious singularities disappear in a universal amplitude combination which may thus be calculated in powers of ϵ as well. To obtain the nonasymptotic critical behavior, however, we need to consider explicitly the critical amplitudes and thus also the way the singularities located at $\epsilon=2/k$ are to be circumvented. This is why the massive field theory was used in I instead of the μ-renormalization scheme. As shown by Parisi[2] and as recalled in I, the massive theory avoids the poles at $\epsilon=2/k$. The drawback is the necessity for a nonperturbative treatment [as shown by Eq. (2.8) for example]. This nonperturbative character is a consequence[24] of the spurious singularities in question. The reason why critical exponents are not affected by these singularities is that they can only be affected by uv divergences at $\epsilon=0$. For example, the calculations of Ref. 28, used by Le Guillou and Zinn-Justin (Ref. 27), consist simply in collecting the poles at $\epsilon=0$ of the ϕ^4 theory inside renormalization functions (similar to the Z_i's of the present paper). This is sufficient to reach the critical singularities (critical exponents). The other singularities located at $\epsilon=2/k$ are not important. *To get the critical singularities, only the structure of the uv divergences at $d=4$ is important.*

Up to now the only method used to consider the ordered phase was in the μ-renormalization scheme which offered the possibility to go continuously from above to below T_c. However, this presents the difficulties discussed just above. We shall now show that it is also possible to use the massive field theory to study the ordered phase. We shall especially obtain the expression of $t_-(g)$.

C. The ordered-phase case ($T<T_c$)

The critical isochore of the ordered phase is characterized by a nonzero spontaneous magnetization $M_0^s(t)$. The correlation functions will depend on the temperature through r_0' and M_0^s; they will be noted $\Gamma_{0,-}^{(L,N)}(\{p\};r_0',g_0,M_0^s)$. Their usual perturbative expansion (in powers of g_0) generates an infinite series of tadpoles.[15] This problem is solved by using instead the loop expansion framework.[16] The general expression of the $\Gamma_{0,-}^{(L,N)}$ may thus be obtained in terms of the correlation functions $\Gamma_{0,+}^{(L,N)}$ of the disordered phase (see following section),

$$\Gamma_{0,-}^{(L,N)}(\{p\};r_0',g_0,M_0)=\sum_{l=0}^{\infty}\frac{(M_0)^l}{l!}\Gamma_{0,+}^{(L,N+l)}(\{p,0\};r_0'+X_0,g_0)\ , \tag{2.9a}$$

$$X_0=g_0M_0^2/2\ . \tag{2.9b}$$

In this paper, M_0 is not necessarily equal to M_0^s, which will be introduced below. A possible renormalization scheme could be strictly copied from the disordered-phase case by using the $\Gamma_{0,-}^{(L,N)}$ in Eqs. (2.2) to define new renormalization functions Z_i. This, of course, would necessitate a recalculation of these functions in terms of a new renormalized coupling constant whose own fixed point value would have to be determined again. In fact these new functions Z_i would display the same singularities (controlled by the same exponents), at the new fixed point, as those of the disordered phase case at g^*. Moreover, a resummation of the power series as straightforward as that above T_c would not at all be guaranteed inasmuch as the series would be shorter (fifth loop instead of sixth loop).

Indeed, it is known that it is sufficient[16] to renormalize the symmetric theory ($T>T_c$) to get a finite (at $d=4$ and infinite cutoff) spontaneous symmetry breaking theory ($T<T_c$). Hence it should be possible to maintain the same $Z_i(g)$ in the two phases. This is what we shall now show.

From consideration of power counting such as is usual in the renormalization techniques of field theory,[16] we can easily verify that the terms $l>0$ of Eqs. (2.9) will not introduce new primitive uv divergences at $d=4$ compared to the case $l=0$ (i.e., $M_0=0$). Hence we can define the new Z_i by Eqs. (2.2) in which r_0' is replaced by $r_0'+X_0$. Although this simplification restores the reference to the symmetric theory we do not yet have a complete similarity between the definitions of the Z_i's in the two phases. This is because, contrary to r_0', X_0 is not an analytic function of t. Indeed, X_0 behaves as $|t|^\beta$. From the known singular behaviors of the Z_i's at the fixed point and their relations with the critical singularities,[10] it is straightforward to verify that at the critical point the combination X_0Z_2/Z_1 vanishes linearly in t.

Let us now write

$$\Gamma_{0,+}^{(L,N)}(\{p\};r_0'+X_0,g_0)=\sum_{l=0}^{\infty}\frac{[(1-Z_2/Z_1)X_0]^l}{l!}\Gamma_{0,+}^{(L+l,N)}(\{p,0\};r_0'+X_0Z_2/Z_1,g_0) \tag{2.10}$$

By using the same argument of power counting as above, we can restrict the definition of the new Z_i's to the part $l=0$ of Eq. (2.10). Let us note that an expansion of $\Gamma_{0,+}^{(L,N)}(\{p\};r_0'+X_0,g_0)$ in powers of X_0 would also solve the problem of the nonlinearity in t but would compromise the loop expansion framework because of the reintroduction of the tadpole difficulty. Instead, the expansion in powers of $(1-Z_2/Z_1)X_0$ in Eq. (2.10) is compatible with the loop expansion framework.

We are now in a position to write down the renormalization scheme for the ordered phase. For the sake of clarity, we shall temporarily note by $\tilde{m}$ and $\tilde{g}$ the new renormalized mass and coupling constant. In terms of these quantities, Eqs. (2.1) remain unchanged while Eqs. (2.2) become

$$[Z_3(\tilde{g})]^{-1}=\frac{\partial}{\partial p^2}\Gamma_{0,+}^{(0,2)}(\{p\};r_0'+X_0Z_2(\tilde{g})/Z_1(\tilde{g}),g_0)|_{p^2=0}\ , \tag{2.11a}$$

$$[Z_1(\tilde{g})]^{-1}=\Gamma_{0,+}^{(0,4)}(\{0\};r_0'+X_0Z_2(\tilde{g})/Z_1(\tilde{g}),g_0)/g_0\ , \tag{2.11b}$$

$$[Z_2(\tilde{g})]^{-1}=\Gamma_{0,+}^{(1,2)}(\{0\};r_0'+X_0Z_2(\tilde{g})/Z_1(\tilde{g}),g_0)\ . \tag{2.11c}$$

In addition, Eqs. (2.3) and (2.4) now read

$$r_0'+X_0Z_2(\tilde{g})/Z_1(\tilde{g})=\tilde{m}^{\,2}/Z_3(\tilde{g})+\delta\tilde{m}^{\,2} \tag{2.12}$$

and

$$\Gamma_{0,+}^{(0,2)}(\{0\};r_0'+X_0Z_2(\tilde{g})/Z_1(\tilde{g}),g_0)=\tilde{m}^{\,2}/Z_3(\tilde{g})\ . \tag{2.13}$$

One may easily verify that, for $g=\tilde{g}$, $Z_i(g)\equiv Z_i(\tilde{g})$ and $m=\tilde{m}$. Indeed, Eqs. (2.11) are simply Eqs. (2.2) in which r_0 is replaced by $r_0'+X_0Z_2/Z_3$. These two quantities are, from Eqs. (2.3) and (2.4) and Eqs. (2.12) and (2.13), the same functions of, respectively, m and $\tilde{m}$. Now, since g_0 is physically fixed, the Z_i's are unchanged for $g=\tilde{g}$ and consequently Eq. (2.1b) implies that $m=\tilde{m}$.

The renormalized coupling constant g is a dummy variable which serves to define the approach to the critical point as it goes to g^*. In the final expression of the critical behaviors it must disappear as the explicit variable and be replaced by t. This is why we shall no longer use tilted quantities (when no confusion can arise). Let us however emphasize that this time *m will no longer be the inverse of the physical correlation length,* although it will display the same kind (same exponent) of singularity at the critical point as it does above T_c. This difference in the physical meaning of m, between I and the present work, is the reason why supplementary Feynman integrals have to be calculated to reach the correlation length of the ordered phase. As was mentioned in the Introduction, this has not been done although it should be quite possible.

It now remains to obtain the expression of t_- as function of g, as in the disordered phase. As in I, we define

$$t_-=r_0'-r_{0c}\ , \tag{2.14}$$

but the two functions $t_-(g)$ and $t_+(g)$ will be different. Following the same steps as were followed to obtain $t_+(g)$, Eqs. (2.8), (2.11c), and (2.13) for $\tilde{g}=g$ and Eq. (2.1b) lead to

$$\frac{d(r_0/g_0^2)}{dg}=Z_2\frac{d}{dg}\left[\frac{[Z_3(g)]^3}{[gZ_1(g)]}\right]-\frac{\partial}{\partial g}\left[X_0\frac{Z_2(g)}{Z_1(g)}\right]\Bigg|_{g_0}\ , \tag{2.15}$$

which finally gives the desired result for the dimensionless scaling field $t_-^*=t_-/g_0^2$:

$$t_-^*(g)=-\int_g^{g^*}dx\left[Z_2(x)\frac{d}{dx}\left[\frac{[Z_3(x)]^3}{[xZ_1(x)]^2}\right][1-U(x)]\right]\ , \tag{2.16}$$

with

$$U(g)=\frac{d}{dg}\left[\frac{X_0^sZ_2(g)}{g_0^2Z_1(g)}\right]\left[Z_2(g)\frac{d}{dg}\left[\frac{[Z_3(g)]^3}{[gZ_1(g)]^2}\right]\right]^{-1}\ . \tag{2.17}$$

Of course X_0^s is related, through Eq. (2.9b), to the M_0^s solution of the equation of state at zero external field,

$$\Gamma_{0,-}^{(0,1)}(r_0',g_0,M_0^s)=0\ . \tag{2.18}$$

X_0^s is a function of r_0' and g_0 and thus, by using the renormalization transformations, it is a function of g and g_0. From dimensional analysis we easily verify that X_0^s/g_0^2 depends only on g.

The function $U(g)$ defined in Eq. (2.17) is nonsingular at the fixed point. By construction, $X_0^sZ_2/g_0^2Z_1$ *vanishes linearly* with t_-^*. Hence $U(g)$, as $g\to g^*$, is proportional to dt_-^*/dg divided by dt_+^*/dg [see also Eq. (2.8)]. Consequently, $U(g^*)=\text{const}$.

By comparing Eqs. (2.8) and (2.16) we see that, at a given value of g close to g^*, the two scaling fields t_+ and t_- are proportional,

$$t_-^*=[1-U(g^*)]t_+^*\ . \tag{2.19}$$

From the results given in Secs. III and IV we can verify that $U(g^*)$ is greater than one, giving t_-^* and t_+^* of opposite signs as expected.

To summarize this section, we have shown that, by keeping unchanged the renormalization scheme for the symmetric theory ($T>T_c$), the calculations below T_c may be done in the massive field theory directly at $d=3$ (Ref. 2) in a way very similar to the one followed in I. In particular, the critical singularities will be expressed via the same fundamental functions of g (the Z_i's), only the critical amplitudes will be modified. New regular functions at g^*, such as $U(g)$, have to be calculated as powers series to get these amplitudes.

It is the object of the following sections to calculate and resum these series.

III. PERTURBATION THEORY

The object of this section is to obtain at $d=3$ the expansions, up to fifth-loop order, in powers of the renormalized coupling constant g, for the spontaneous magnetization M_0^s, the susceptibility χ^-, and the specific heat C^- along the critical isochore below T_c. The starting point is the usual perturbation theory for a ϕ^4 Hamiltonian [Eq. (1.1)] with Λ infinite but dimensionally regularized ($d=4-\epsilon$) (see I and Sec. II).

The quantities we are interested in may be all derived from the knowledge of the free energy $\Gamma_{0,-}^{(0,0)}(r_0,g_0,M_0,\epsilon)$.[29] Its expansion up to fifth-loop order reads

$$\Gamma^{(0,0)}_{0,-}(r_0,g_0,M_0,\epsilon)=\tfrac{1}{2}r_0M_0^2+g_0\frac{M_0^2}{24}+\sum_{b=1}^{5}\Gamma_b(g_0,r_0,M_0,\epsilon)\ . \tag{3.1}$$

At a given number b of loops, Γ_b is expressed as a sum over Feynman integrals corresponding to the various connected (one-particle irreducible) graphs, with b loops and zero external leg, that can be drawn with three- and four-leg vertices.[15] This sum may be written as follows:

$$\Gamma_b(r_0,g_0,M_0,\epsilon)=-(-A_d')^b g_0^{d/\epsilon}\tilde{r}_0^{[d-\epsilon(b-1)]/2}\sum_{l=0}^{b-1}\sum_{m=1}^{i}(-2\tilde{X}_0)^l P_m^{(b,l)} I_m^{(b,l)}(\epsilon)\ , \tag{3.2}$$

with dimensionless $\tilde{r}_0$ and $\tilde{X}_0$ defined by

$$\tilde{r}_0=(r_0+X_0)/g_0^{2/\epsilon}\ , \tag{3.3}$$

$$\tilde{X}_0=X_0/(r_0+X_0)\ . \tag{3.4}$$

Each $I_m^{(b,l)}(\epsilon)$ is a dimensionless Feynman integral at d dimensions associated to the mth of the $i(b,l)$ connected graphs that can be drawn with b loops and $2l$ three-leg vertices, $P_m^{(b,l)}$ is the weight factor associated to this graph. For convenience,[14] a geometrical factor A_d' has been factorized out with $A_d'=\Gamma_E(2-d/2)/(4\pi)^{d/2}$.

A. Renormalization at $d=3$

In order to calculate the integrals $I_m^{(b,l)}$ at $d=3$, a subtraction of the poles at $\epsilon=1$ must first be defined. As already mentioned, *3-d renormalization does not concern the critical singularities.* It corresponds to the subtraction of the primitive uv divergences which, at $d=4$, are higher than the logarithmic divergences. In I, it is shown that only $\Gamma_0^{(0,0)}$, $\Gamma_0^{(1,0)}$, and $\Gamma_0^{(0,2)}$ have to be considered[30] and the latter case is explicitly treated. This led to the definition of r_0' introduced in Sec. II B of the present paper (see also Appendix A). The two other kinds of subtraction correspond to the definition of two arbitrary constants to be added to the free energy and its derivative with respect to temperature. As mentioned in Sec. II, this arbitrariness expresses the impossibility of describing in RG theory the noncritical (regular) part ($\mathcal{F}_{\rm reg}$) of the free energy $\mathcal{F}$. Indeed, $\mathcal{F}_{\rm reg}$ may be written under the form $\mathcal{F}_0+\mathcal{F}_1T$. The two arbitrary constants $\mathcal{F}_0$ and $\mathcal{F}_1$ correspond to the fact that the internal energy and the entropy of a system can never be absolutely measured. Since the free energy serves, in particular, to define the specific heat, which does not contain $\mathcal{F}_{\rm reg}$, we shall fix this regular part to be zero. Hence, $\mathcal{F}$ will reduce to its singular part $\mathcal{F}_{\rm sing}$.[25] This condition is sufficient to determine the counterterm at $d=3$. Let us look at this point in detail.

The definition of $\mathcal{F}_{\rm sing}$ is the primitive with respect to T of the singular part of the specific heat $C_{\rm sing}=-\Gamma_0^{(2,0)}$. Once expressed in terms of r_0', $\Gamma_0^{(2,0)}$ only displays, at $d=4$, logarithmic uv divergences and is no longer affected by 3-d renormalization. It is likely that $\mathcal{F}_{\rm sing}$ does not need such a renormalization and is finite at $d=3$ (and Λ infinite). Thus if we reduce the free energy to its singular part ($\mathcal{F}_{\rm reg}=0$), it is equivalent to having performed 3-d renormalization. Consequently, calculation of a Feynman integral $I_m^{(b,l)}(\epsilon=1)$ is straightforward: either it converges and it is directly calculable, or it diverges and its subtracted value is obtained by calculating the primitives of its derivatives with respect to r_0' as is shown by the following example (see also Appendix A).

Let us consider the Feynman integral $I_1^{(4,0)}$ relative to the graph drawn in Fig. 1(a). This integral diverges at $d=3$. Let us define, according to Eq. (3.2),

$$I(\tilde{r}_0,\epsilon)=\tilde{r}_0^{2(1-\epsilon)}I_1^{(4,0)}(\epsilon) \tag{3.5}$$

and consider the Feynman integral $J_1(\epsilon)$ associated respectively to the convergent graph drawn in Fig. 1(b). Let us define

$$J(\tilde{r}_0,\epsilon)=\tilde{r}_0^{(1-2\epsilon)}J_1(\epsilon)\ . \tag{3.6}$$

From the Feynman rules we easily verify that

$$\frac{d}{d\tilde{r}_0}I(\tilde{r}_0,\epsilon)=-6J(\tilde{r}_0,\epsilon)\ . \tag{3.7}$$

Now $J_1(1)$ is convergent, thus at $d=3$, from Eqs. (3.6) and (3.7),

$$\frac{d}{d\tilde{r}_0}I(\tilde{r}_0,1)=-\frac{6}{\tilde{r}_0}J_1(1)\ . \tag{3.8}$$

Hence the primitive with respect to $\tilde{r}_0$ gives

$$I(\tilde{r}_0,1)=-6J_1(1)\ln(\tilde{r}_0)\ , \tag{3.9}$$

which will be the 3-d renormalized value of the Feynman integral $I_1^{(4,0)}$ associated to the graph drawn in Fig. 1(a) and displayed in Table I.

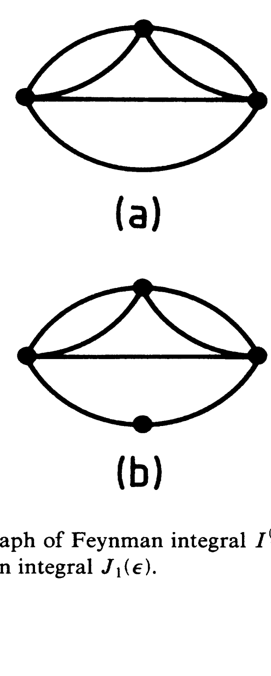

FIG. 1. (a) Graph of Feynman integral $I_1^{(4,0)}$. (b) Convergent graph of Feynman integral $J_1(\epsilon)$.

In Appendix A, we show how the case of subtraction related to the mass shift is treated.

The striking effect of 3-d renormalization is the appearance of logarithms of $\tilde{r}_0$, and thus of the bare coupling constant g_0 as predicted by Symanzik.[24]

In Table I, we present the values of the Feynman integrals at $d=3$ (after 3-d renormalization) associated to all one-vertex irreducible graphs up to fifth-loop order. The values of the other integrals contributing to $\Gamma_{0,-}^{(0,0)}$ may be obtained as simple products of one-vertex irreducible Feynman integrals. The complete explicit expression of $\Gamma_{0,-}^{(0,0)}$ in terms of the integrals of Table I is given in Appendix B.

B. Perturbative expressions of M_0^s, χ^-, and C^-

In principle, these quantities are obtained from derivatives of the free energy $\mathscr{F}(T,M_0)$. The spontaneous magnetization is the nontrivial solution of the equation

$$\left[\left.\frac{\partial\mathscr{F}(T,M_0)}{\partial M_0}\right|_T\right]_{M_0=M_0^s(T)}=0\ , \tag{3.10}$$

giving $M_0^s(T)$. The susceptibility and the specific heat along the critical isochore are respectively defined by

$$1/\chi^-(T)=\left[\left.\frac{\partial^2\mathscr{F}(T,M_0)}{\partial M_0^2}\right|_T\right]_{M_0=M_0^s(T)}\ , \tag{3.11}$$

$$C^-(T)=-T\frac{d^2\mathscr{F}[T,M_0^s(T)]}{dT^2}\ . \tag{3.12}$$

Since $\mathscr{F}_{\rm reg}$ depends only on T, the definitions of M_0^s and $\chi^-(T)$ are unchanged if one replaces $\mathscr{F}$ by $\mathscr{F}_{\rm sing}$ in Eqs. (3.10) and (3.11). In other words, 3-d renormalization does not influence the results (apart from the mass shift). As for $C^-(T)$ we have to consider separately the regular and singular parts. As in I we write

$$C^-(T)=C_{\rm sing}^-(T)+B_{\rm bg}(T)\ , \tag{3.13}$$

with,

$$B_{\rm bg}(T)=-T\frac{d^2\mathscr{F}_{\rm reg}(T)}{dT^2}\ . \tag{3.14}$$

Following what has been said above on the meaning of $\mathscr{F}_{\rm reg}$, we must have $B_{\rm bg}\equiv 0$ as far as only criticality is concerned. That this regular part of the specific heat is not present in the critical behavior of C is not surprising. Indeed, as is strongly stated in Ref. 31, the study of critical phenomena requires taking into account all wavelength scales of fluctuations between the atomic spacing and the correlation length,[8] and all these scales contribute the same to the critical singularities. In the framework of FT this is conveyed by logarithmic uv divergences. In particular the logarithmic uv divergence of the specific heat, additively subtracted, entirely generates the usual additive constant in the critical behavior of C. There is no remaining fluctuation to be considered which could explain the presence of an extra term like $B_{\rm bg}$ which is related to nonlogarithmic divergences (definition of $\mathscr{F}_{\rm reg}$, see Sec. VI).

Neglecting analytic corrections to scaling (see Sec. VI), we finally have the following definitions: M_0^s is the solution of the equation

$$\Gamma_{0,-}^{(0,1)}(r_0',g_0,M_0^s)=0\ , \tag{3.15}$$

while the susceptibility and the singular part of C^- are given by

$$1/\chi^-=\Gamma_{0,-}^{(0,2)}(r_0',g_0,M_0^s)\ , \tag{3.16}$$

$$C_{\rm sing}^-=-\Gamma_{0,-}^{(2,0)}(r_0',g_0,M_0^s)\ . \tag{3.17}$$

In these equations, the correlation functions $\Gamma_{0,-}^{(L,N)}$ are 3-d renormalized as indicated by the replacement of r_0 by r_0' and by implicitly setting ϵ equal to 1.

In the following, the dimensionless quantities $\tilde{r}_0$ and $\tilde{X}_0$ will be defined by Eqs. (3.3) and (3.4) in which $\epsilon=1$ and r_0 is replaced by r_0'. For the sake of notational convenience, we shall use in the following $\tilde{g}_0=(\tilde{r}_0)^{-1/2}$ as the dimensionless quantity instead of $\tilde{r}_0$.

As shown above, the perturbative expansion of the $\Gamma_{0,-}^{(L,N)}$ will, in addition to powers of $\tilde{g}_0$ and of $\tilde{X}_0$, contain powers of logarithms of $\tilde{g}_0$. Hence the general form of $\Gamma_{0,-}^{(0,0)}$ is

$$\Gamma_{0,-}^{(0,0)}(r_0',g_0,M_0)=\tfrac{1}{2}r_0'M_0^2+g_0\frac{M_0^4}{24}+(r_0'+X_0)^{3/2}\sum_{b=1}^{5}\sum_{l=0}^{b-1}\sum_{k=0}^{2}F_{blk}\tilde{g}_0^{(b-1)}\tilde{X}_0^{l}(\ln\tilde{g}_0)^k \tag{3.18}$$

The coefficients F_{blk} obtained from Table I and Eqs. (3.1) and (3.2) after 3-d renormalization are displayed in Table II.

Similarly, we can write the following: (i) for the equation of state

$$\Gamma_{0,-}^{(0,1)}(r_0',g_0,M_0)=M_0\left[r_0'+\frac{X_0}{3}+(r_0'+X_0)\sum_{b=1}^{5}\sum_{l=0}^{b-1}\sum_{k=0}^{2}H_{blk}\tilde{g}_0^{(b-1)}\tilde{X}_0^{l}(\ln\tilde{g}_0)^k\right]\ , \tag{3.19}$$

with

$$H_{blk}=\left[2-\frac{b}{2}-l\right]F_{blk}+2(l+1)F_{b,l-1,k}+(k+1)F_{b,l,k+1}\ . \tag{3.20}$$

The nontrivial zero of the right-hand side of Eq. (3.19) gives M_0^s as a function of T (through r_0'). (ii) For the susceptibility

$$\Gamma_{0,-}^{(0,2)}(r_0',g_0,M_0)=r_0'+\frac{X_0}{2}+\sum_{b=1}^{5}\sum_{l=0}^{b}\sum_{k=0}^{2}Q_{blk}\tilde{g}_0^{b}\tilde{X}_0^{l}(\ln\tilde{g}_0)^k\ , \tag{3.21}$$

TABLE II. Coefficients F_{blk} of the singular part of the free energy below T_c [Eq. (3.18)] up to fifth-loop order.

		F_{blk}		
b	k/l	0	1	2
1	0	$-0.265\,258\,238\times 10^{-1}$		
2	0	$0.791\,571\,747\times 10^{-3}$		
	1		$-0.105\,542\,900\times 10^{-2}$	
3	0	$0.141\,653\,916\times 10^{-4}$	$0.419\,941\,855\times 10^{-4}$	
	1	$0.436\,866\,234\times 10^{-4}$		
	2	$-0.100\,056\,131\times 10^{-4}$		
4	0	$0.449\,205\,291\times 10^{-6}$	$0.122\,593\,698\times 10^{-5}$	
	1	$-0.115\,567\,060\times 10^{-5}$	$-0.556\,965\,183\times 10^{-6}$	
	2	$0.100\,772\,569\times 10^{-5}$		
	3	$-0.300\,586\,776\times 10^{-6}$		
5	0	$-0.384\,004\,055\times 10^{-7}$	$0.146\,856\,580\times 10^{-7}$	$-0.110\,804\,703\times 10^{-7}$
	1	$0.920\,659\,875\times 10^{-7}$	$0.230\,540\,645\times 10^{-7}$	
	2	$-0.100\,671\,980\times 10^{-6}$	$-0.158\,403\,213\times 10^{-7}$	
	3	$0.647\,972\,392\times 10^{-7}$		
	4	$-0.165\,599\,191\times 10^{-7}$		

with

$$Q_{blk}=\left[2-\frac{b}{2}-l\right]H_{b,l-1,k}+(1+2l)H_{blk}+(k+1)H_{b,l-1,k+1}\,. \tag{3.22}$$

To get the critical isochore expression of χ^- we must replace M_0 by M_0^s in Eq. (3.21). (iii) For the specific heat the definition mixes derivatives with respect to M_0 and to r_0'. We obtain

$$C_{\rm sing}^-(r_0',g_0,M_0^s)=-\frac{1}{g_0}\left[C_1(\tilde{g}_0,\tilde{X}_0^s)+2\frac{dX_0^s}{dr_0'}C_2(\tilde{g}_0,\tilde{X}_0^s)+\left[\frac{dX_0^s}{dr_0'}\right]^2\frac{[\chi(r_0',g_0,M_0^s)]^{-1}}{2X_0^s}\right], \tag{3.23}$$

with

$$C_1(\tilde{g}_0,\tilde{X}_0^s)=g_0\Gamma_{0,-}^{(2,0)}(r_0',g_0,M_0^s) \tag{3.24}$$

and

$$C_2(\tilde{g}_0,\tilde{X}_0^s)=\Gamma_{0,-}^{(1,1)}(r_0',g_0,M_0^s)/M_0^s\,. \tag{3.25}$$

The set of Eqs. (3.19)–(3.25) gives the perturbative expressions of the physical quantities of interest. *Their critical behaviors have still to be obtained.* This must be done by considering the renormalization process which subtracts the logarithmic uv divergences obtained from the four-dimensional theory.

C. Renormalization at $d=4$

Following Sec. II, this renormalization consists in performing on the perturbative expressions obtained in the previous part, the following change of variables:

$$g_0=mgZ_1(g)/[Z_3(g)]^2\,, \tag{3.26}$$

$$r_0'+X_0=m^2/Z_3(g)+X_0[1-Z_2(g)/Z_1(g)]+\delta\overline{m}^2\,, \tag{3.27}$$

in which the Z_i's are powers series in g already known up to sixth-loop order.[14] The mass shift $\delta\overline{m}^2$ is defined by Eq. (2.13) with $\tilde{g}=g$ and $\tilde{m}=m$. More precisely, we have

$$\delta\overline{m}^2=r_0'+X_0-\Gamma_{0,+}^{(0,2)}(\{0\};r_0'+X_0,g_0)\,, \tag{3.28}$$

which, by using Eqs. (3.26) and (3.27), gives $\delta\overline{m}^2$ as a power series in g.

Reporting these changes of variables inside Eq. (3.19) and solving $\Gamma_{0,-}^{(0,1)}=0$, we obtain X_0^s as a power series in g, its dependence on g_0 being obtained from simple dimensional analysis once m is reexpressed, via Eq. (3.26), as function of g_0 and g. Notice that the logarithms of g_0 disappear in $\Gamma_{0,-}^{(0,1)}$ after the shifting of mass of Eq. (3.27). This is also the case for the susceptibility and the specific heat when the same change of variables is performed inside Eqs. (3.21) and (3.23).

Finally, we obtain all the physical quantities of interest as power series in g. It is important to note that they are the bare quantities expressed in terms of the renormalized coupling constant g and dimensioned by powers of g_0. They will display singularities at g^* which can be expressed through a factorization of powers of the Z_i's. The following section presents this step in detail.

D. Critical singularities of M_0^s, χ, and C

1. Expression for M_0^s

The critical singularity of M_0^s is in $|t|^\beta$. Since $|t|$ vanishes as $|g-g^*|^{1/\Delta}$ with $\Delta=\omega\nu$, M_0^s will present a singularity at g^* of the form $|g-g^*|^{\beta/\Delta}$. From the scaling law $\beta=\nu(1+\eta)/2$ (at $d=3$) one easily obtains $M_0^s \simeq |g-g^*|^{(1+\eta)/2\omega}$. From the known behaviors[10] of the Z_i's near g^*, this singularity may be expressed via the combination $\{[Z_3(g)]^3/[gZ_1(g)]\}^{1/2}$. Hence one can define, from $X_0^s(g)$, a nonsingular series, noted $X(g)$, as follows:

$$X_0^s(g)=g_0^2\frac{[Z_3(g)]^3}{gZ_1(g)}\frac{X(g)}{g} . \tag{3.29}$$

2. Expression for χ^-

Following similar argument, we define a nonsingular series $S(g)$ from $\chi^-(g)$ by writing

$$1/\chi^-(g)=g_0^2\frac{[Z_3(g)]^3}{[gZ_1(g)]^2}S(g) . \tag{3.30}$$

3. The specific-heat case

The singular part of the specific heat contains, in addition to a singularity at $T=T_c$, a critical constant B_{cr}.[10,18] So, the factorization of the Z_i's could be obtained, as in I, from its derivative with respect to t. However, this procedure shortens the series by one term. We shall, thus, use the fact that B_{cr} is the same in the two phases and define the quantity

$$\Delta C(g)=C^-(g)-C^+(g) , \tag{3.31}$$

in which the singularity at T_c may be factorized out

$$\Delta C(g)=\frac{gZ_1(g)}{[Z_2(g)]^2}\widetilde{F}(g)/(8\pi g_0) . \tag{3.32}$$

So defined, $\widetilde{F}(g)$ has no singularity at $g=g^*$. The factor $1/8\pi$ is introduced for convenience.

E. Universal combinations of leading critical amplitudes

In the above considerations no difference was made between the two phases in the definition of the renormalized coupling constant g. Indeed g, in the approach to the critical point from above or below, plays the role of a dummy variable. We can thus allow g to have the same value in the two phases and to compare the physical quantities of interest for this value of g. As already stated (see Sec. II), the Z_i's are the same functions of g above and below T_c. Since they implicitly involve the critical singularities (as g approaches g^*), they will disappear in a universal amplitude combination corresponding to a scaling-law relation between critical exponents.

For example, let us consider the susceptibilities in the two phases expressed as functions of the same g. We obtain, from Eq. (3.30) and Eq. (4.3b) of I,

$$\chi^+(g)/\chi^-(g)=S(g) . \tag{3.33}$$

However, this ratio does not give the universal ratio Γ^+/Γ^- with $\Gamma^\pm$ defined by

$$\chi^\pm(t)\underset{t\to 0}{\simeq}\Gamma^\pm|t|^{-\gamma} \tag{3.34}$$

in which $|t|$ has the same value above and below T_c. Indeed, for a given g, we have [see Eq. (2.19)]

$$t_-(g)\underset{g\to g^*}{\simeq}[1-U(g^*)]t_+(g) . \tag{3.35}$$

From Eqs. (3.33)—(3.35) we thus obtain

$$\Gamma^+/\Gamma^-=S(g^*)/|1-U(g^*)|^\gamma . \tag{3.36}$$

It is left now to sum, at $g=g^*$, the nonsingular series $S(g)$ and $U(g)$ to obtain an estimate, at $d=3$, of the ratio Γ^+/Γ^-. The other universal combinations of leading critical amplitudes are obtained using the same line of reasoning. The object of the following section is to provide the technical detail concerning this obtainment and also to present the resummation of the corresponding series.

IV. UNIVERSAL COMBINATIONS OF LEADING CRITICAL AMPLITUDES. RESUMMATION OF THE SERIES

As shown just above, the universal amplitude combinations can be expressed as combinations of nonsingular series at g^*, such as $S(g)$, $X(g)$, $\widetilde{F}(g)$, and $U(g)$, to be resummed at $g=g^*$. The aim of this section is to present the resummation procedure which we have used. Before looking at the resummation itself, let us emphasize simplifications occurring at $g=g^*$ in the expressions of interest.

A. Simplifications at $g=g^*$

Let us consider the definition of $U(g)$ given by Eq. (2.17). From the definition of the Wilson function $W(g)$ and of the critical exponent functions $\nu(g)$ and $\gamma(g)$ (see I) we can easily verify that $U(g)$ may be written as follows:

$$U(g)=\frac{X(g)}{\gamma(g)}\left[1+W(g)\nu(g)\frac{d}{dg}\ln X(g)\right] . \tag{4.1}$$

From the definition of g^* [such that $W(g^*)=0$], we immediately obtain,

$$U(g^*)=\frac{X(g^*)}{\gamma} , \tag{4.2}$$

in which γ stands for $\gamma(g^*)$.

This kind of simplification will now be used to obtain the expression for the universal amplitude combinations [except for Γ^+/Γ^- which is given by Eq. (3.36)].

B. Expression of the specific-heat amplitude ratio A^+/A^-

The critical constant B_{cr} of the specific heat prevents us from directly obtaining A^+/A^-. The way we have obtained this ratio is as follows.

Since B_{cr} is the same above and below T_c, we have for $g=g^*$

$$\Delta C\simeq\frac{1}{\alpha}[A^-|1-U(g^*)|^{-\alpha}-A^+](t_+)^{-\alpha} , \tag{4.3}$$

in which ΔC is defined in Eq. (3.31). We also have

$$\frac{\partial C^+}{\partial t_+}=-A^+(t_+)^{-(\alpha+1)} . \tag{4.4}$$

The ratio $\mathscr{R}$ of these two quantities varies linearly with t_+:

$$\mathscr{R} \equiv \Delta C \left[\frac{\partial C^+}{\partial t_+} \right]^{-1} = \frac{A^+ - A^- \,|\, 1 - U(g^*) \,|^{-\alpha}}{\alpha A^+} t_+ \ . \tag{4.5}$$

Hence we obtain

$$A^-/A^+ = |\, 1 - U(g^*) \,|^{\alpha} [1 - \alpha(d\mathscr{R}/dt_+)] \ . \tag{4.6}$$

The quantity $\mathscr{R}$ is a function of g. From the definitions of ΔC [Eqs. (3.31) and (3.32)], $\partial C^+/\partial t_+$ [Eqs. (4.20) of I] and dt_+/dg [Eq. (2.8)], and using the same line of arguments leading to Eq. (4.1), we can easily obtain the following result:

$$\frac{d\mathscr{R}(g)}{dt_+} = \frac{\widetilde{F}(g)}{\gamma(g)F(g)} \left[1 + W(g)\nu(g) \frac{d}{dg} \ln[\widetilde{F}(g)/F(g)] \right] . \tag{4.7}$$

The series $F(g)$ are defined in I. At $g = g^*$, we obtain

$$\frac{d\mathscr{R}}{dt_+} = \frac{\widetilde{F}(g^*)}{\gamma F(g^*)} \ . \tag{4.8}$$

Finally, from Eq. (4.6), we obtain

$$A^-/A^+ = |\, 1 - U(g^*) \,|^{\alpha} \left[1 - \frac{\alpha \widetilde{F}(g^*)}{\gamma F(g^*)} \right] , \tag{4.9}$$

which we have used to estimate A^+/A^-. In this expression $U(g^*)$ may be obtained via Eq. (4.2).

C. Expression of the combination R_C^+

This universal combination is defined as follows:[32,33]

$$R_C^+ = A^+ \Gamma^+ / B^2 \ , \tag{4.10}$$

in which B is the critical amplitude of the spontaneous magnetization,

$$M_0^s \simeq B \,|\, t_- \,|^{\beta} \ . \tag{4.11}$$

As in the case of A^+/A^-, R_C^+ cannot be obtained from the simple combination $\mathscr{R}'(t)$ defined by

$$\mathscr{R}'(t) = \frac{M_0^s}{(\partial C^+/\partial t_+)\chi^+} \ . \tag{4.12}$$

Instead we have, for g near g^*,

$$\mathscr{R}'(t) = -\frac{|\, 1 - U(g^*) \,|^{2\beta}}{R_C^+} (t_+)^3 \ . \tag{4.13}$$

This quantity $\mathscr{R}'$ is an implicit function of t through its dependence on g. From the same kind of arguments as in the above two subsections, we easily obtain

$$R_C^+ = \frac{\gamma^2 g^*}{16\pi} \frac{F(g^*)}{U(g^*)} \,|\, 1 - U(g^*) \,|^{2\beta} \ . \tag{4.14}$$

D. Resummation of the series

As is clearly seen from Eqs. (3.36), (4.2), (4.6), and (4.14), the estimates for the universal amplitude combinations need resummation of only three series: $X(g)$, $S(g)$, and $\widetilde{F}(g)$ at the fixed point g^*. These series are displayed in Table III. The values of the other quantities, such as γ, $F(g^*)$ or g^*, are already known from the work done in I.

The resummation method used is based on the knowledge of the large-order behavior of the series. One knows that the kth order of any of these series, say $f^{(k)}$, behaves as follows:[34,35]

$$f^{(k)} \simeq k!(-a_0)^k k^{b_0} c_0 [1 + O(1/k)] \ . \tag{4.15}$$

We shall not recall here in detail the transformations[4] of the series, suggested by this asymptotic behavior, which allow the resummation (Borel-Loeffel transform and conformal mapping) to be carried out. The essential ideas are based on the following: once each term of the series is divided by $k!$ (Borel transform) the nearest singularity at the origin, located at $-1/a_0$ on the real axis, has a strength controlled by b_0. An analytic continuation outside the circle of convergence, needed to define the inverse Borel transform, is obtained by a conformal mapping. An adjustable parameter b is introduced by replacing the division by $k!$ by the division by $\Gamma_E(k+b+1)$ (Borel-Loeffel transform). This parameter b allows the strength of the singularity (at $-1/a_0$) to be modified. A second adjustable parameter μ is also introduced, which modifies the singularity of the function to be resummed, at a large coupling constant. These two parameters, when varied, modify the convergence of the resummation. In general, there is a couple (b',μ') at which the convergence is the best.[4] Around (b',μ'), oscillating convergences allow the

TABLE III. Power series, in the coupling constant $v = 3g/(16\pi)$, to be resummed. On the first line are zeroth orders (constant terms). We see, for example, on S case, that series for the ordered phase $(X,S,\widetilde{F})$ do not verify the large-order behavior of Eq. (4.15) (the sign does not alternate). Only F of the disordered phase has been resummed using the standard method of Refs. 4 and 36, the other three will be modified before resummation (see text and Table IV).

X	S	$\widetilde{F}$	F
1.5	1.0	4.5	-0.25
0.0	0.0	0.0	0.0
0.0	$-0.166\,666\,666\,6$	$0.333\,333\,333\,1$	$-7.129\,975\,964\times10^{-3}$
$2.992\,853\,488\times10^{-2}$	$2.901\,857\,099\times10^{-2}$	$-0.177\,751\,281\,4$	$3.870\,342\,824\times10^{-3}$
$-2.306\,916\,594\times10^{-2}$	$1.686\,963\,552\times10^{-3}$	$6.898\,803\,732\times10^{-2}$	$-3.294\,284\,423\times10^{-3}$
$-3.049\,803\,545\times10^{-2}$	$0.147\,979\,713\,9$	$-0.269\,320\,907\,4$	$4.125\,325\,803\times10^{-3}$

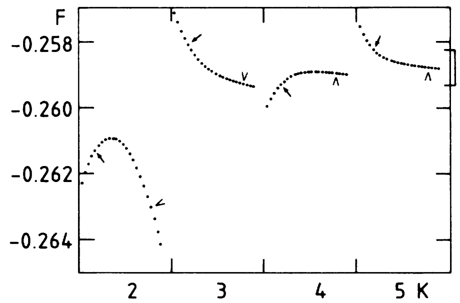

FIG. 2. Evolution of the resummed series $F(v)$ (see Table III) at $v^*=1.416$ [$v=3g/(16\pi)$] as functions of the order K and of the parameter b which is varied from -0.5 to 12.5 by steps of 0.5 (dotted line). The parameter μ is fixed to 2.5 (see text for the definitions of b and μ). The arrows $\rightarrow$ indicate the value $b=1.5$ corresponding to an oscillating convergence. The symbols $>$ indicate, for $b=9.5$, a smooth convergence. The vertical bar —— on the right represents the final estimate of F for $v^*=1.416$ ($-0.258\,74\pm0.000\,53$).

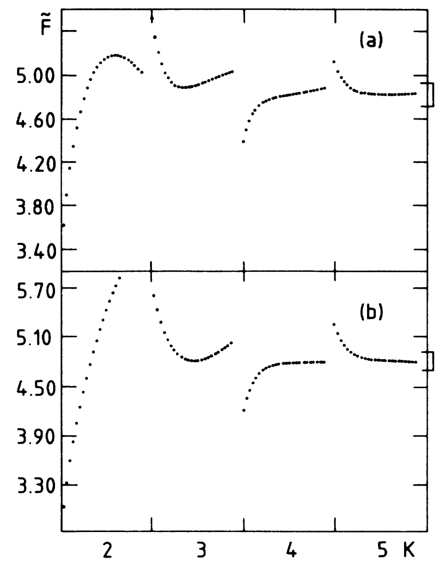

FIG. 3. Evolution of the resummed series $\tilde{F}(v)$ from the series $\tilde{F}'(v)$ of Table IV at $v^*=1.416$. The presentation is the same as for Fig. 2, with (a) $\mu=0.5$, (b) $\mu=1.5$. The final estimate is here $\tilde{F}=4.822\pm0.108$.

error to be estimated.[4,36] In Fig. 2 we illustrate the result of this resummation in the case of series $F(g)$ relative to the disordered phase for $g=g^*$.

In the case of series S, X, and $\tilde{F}$, relative to the ordered phase, the resummation method, applied as described above, does not lead to any clear oscillating convergence. Consequently, it is almost impossible to estimate the error following the scheme briefly recalled above, although convergence could be observed.

The main reasons for this situation come from Eqs. (2.9). Calculation below T_c of $\Gamma_-^{(N)}$ involves a sum over l of $\Gamma_+^{(N+l)}$ with a number of external legs increasing with the order. Now the strength b_0 of the singularity of the series Borel transformed depends on this number of external legs.[35] Furthermore, at a given order in powers of g for $\Gamma_-^{(N)}$, the number of orders of $\Gamma_+^{(N+l)}$ in Eq. (2.9) decreases as l increases. For the highest values of l the large-order regime is clearly not reached for the $\Gamma_+^{(N+l)}$. In spite of this, the large-order behavior as described by Eq. (4.15) remains unchanged and thus the principle of the resummation method will be maintained. However, in order to allow clear oscillating convergences, we have chosen to subtract, from the series to be resummed, a linear combination of known functions of g having a large-order behavior like that of Eq. (4.15) when expanded in powers of g. The functions used are the following:

$$B_{kr}(g)=\int_0^\infty \frac{dx}{x}\frac{e^{-x}x^{k/2}}{(1+a_0gx)^r}\ , \tag{4.16}$$

in which k and r are adjustable parameters (for technical reasons k is limited to integer values), and a_0 is known[4,35] for ϕ^4 theory at $d=3$.

We have then chosen the values of r and of the coefficients C_k such that the combination,

$$\tilde{F}'(g)=\tilde{F}(g)+\sum_{k=1}^{3}C_kB_{kr}(g)\ , \tag{4.17}$$

leads to power series $\tilde{F}'$ whose coefficients $\tilde{F}'_k$ are such that the ratios $\tilde{F}'_{k+1}/\tilde{F}'_k$ for the last three orders, are identical to those relative to series for $1/\gamma(g)$, whose resummation led to clear oscillating convergences.[4] The choice for r was indeed very wide and several cases were tried leading to similar results. In Table IV, we display the primed series, according to Eq. (4.17), with $r=6$, for $X'(g)$, $S'(g)$, and $\tilde{F}'(g)$, which we have resummed instead of $X(g)$, $S(g)$, and $\tilde{F}(g)$. The oscillations observed are shown in Fig. 3 in the case of $\tilde{F}(g^*)$. In spite of great differences between the terms of the series $\tilde{F}(g)$ (Table III) and $\tilde{F}'(g)$ (Table IV) the central value remains the same.

TABLE IV. Modified power series in $v=3g/(16\pi)$ according to Eq. (4.17). The last column displays the series $1/\gamma(v)$ which has been chosen as a model to construct the other three from their expressions of Table III (see text). Same presentation as in Table III.

X'	S'	$\tilde{F}'$	γ^{-1}
$-0.239\,785\,162\,1$	$18.069\,949\,29$	$-47.092\,436\,40$	1.
$0.275\,608\,256\,3$	$-3.687\,398\,280$	$11.910\,664\,54$	$-0.166\,666\,666\,7$
$-6.513\,876\,393\times10^{-2}$	$1.374\,860\,601$	$-5.043\,431\,914$	$3.703\,703\,700\times10^{-2}$
$4.057\,361\,867\times10^{-2}$	$-0.856\,372\,862\,8$	$3.141\,451\,739$	$-2.306\,962\,130\times10^{-2}$
$-3.497\,587\,814\times10^{-2}$	$0.738\,223\,354\,9$	$-2.708\,041\,255$	$1.988\,682\,030\times10^{-2}$
$3.950\,060\,507\times10^{-2}$	$-0.833\,725\,148\,3$	$3.058\,372\,621$	$-2.245\,952\,000\times10^{-2}$

TABLE V. Values, after resummation, of the nonsingular series at the fixed point. The coupling v is $3g/(16\pi)$. Max and Min correspond to an estimate of the error at fixed v. The two values $v^*_{\rm max}$ and $v^*_{\rm min}$ correspond respectively to the upper and lower estimates of the fixed point v^* according to the resummation of $W(v)$. The final errors in the estimates of universal amplitude ratios given in this work (Table VIII) account for these two sources of uncertainty.

Series		$v^*_{\rm max}=1.420215$	$v^*_{\rm min}=1.410942$
$S(v^*)$	Max	0.80588	0.80754
	Min	0.76090	0.76351
$X(v^*)$	Max	1.53795	1.53732
	Min	1.52443	1.52412
$F(v^*)$	Max	−0.25932	−0.25921
	Min	−0.25825	−0.25817
$\tilde{F}(v^*)$	Max	4.9322	4.9269
	Min	4.7138	4.7134

The resummation of $\tilde{F}'(g)$, however, allows an estimate of error following the same rule as in the case of $F(g)$. Figure 3 illustrates the error estimate for $\tilde{F}(g^*)$. From this procedure, $X(g^*)$, $S(g^*)$, and $\tilde{F}(g^*)$ are known, the results are given in Table V. Two possible values of g^* are considered corresponding to the upper (max) and lower (min) bounds of the error analysis[10] relative to the determination of the zero of $W(g)$. At each bound of g^*, the series are resummed leading to an error estimate attached to this bound.

The universal amplitude combinations can now be calculated, using Eqs. (3.36), (4.2), (4.6), and (4.14). The results are displayed in Table VI. The indicated error accounts for the sum of errors in the estimation of each term entering in the corresponding equation, without mixing the two bounds of g^*. In other words, errors are calculated two times for each universal quantity, one at $g^*_{\rm max}$ the other one at $g^*_{\rm min}$. The upper and lower results are retained to give errors shown in Table VI.

By comparing these results with those of Ref. 13 we see that, although compatible, they slightly differ. The main reason comes from the knowledge of one additional order and from a more careful resummation.

Since the resummation method does not depend on g, it is possible to get nonasymptotic critical behaviors of M_0^s, χ^-, and C^- as in I. It is the object of the following section to present such calculations.

V. NONASYMPTOTIC CRITICAL BEHAVIOR AND FIRST CORRECTIONS TO SCALING

A. Summary of the preceding sections and preliminaries

In Secs. III and IV, we have shown how to calculate the perturbative expressions of M_0^s, χ^-, and C^-, at $d=3$, in terms of the renormalized coupling constant g already used in the disordered phase case. We have then shown that the universal leading critical amplitude combinations can be expressed in terms of a few numbers of series, namely $S(g)$, $X(g)$, $\tilde{F}(g)$, and $F(g)$, and of the usual critical exponents. We have resummed these series at the fixed point g^* to get the estimates of A^+/A^-, Γ^+/Γ^-, and R_C^+ displayed in Table VI. The object of this section is to make a nonasymptotic study of the available measurable quantities, similar to that done in I. The first step is to express the critical singularities of the quantities of interest in terms of series having no singularity at g^*. Following Eqs. (4.14) of I and using the results of the preceding sections, we can easily verify that we may write the physical quantities of interest under the following forms:

(i) The spontaneous magnetization [from Eqs. (2.9b) and (3.29)],

$$M_0^s(g)=\left[\frac{2g_0[Z_3(y_0)]^3}{y_0Z_1(y_0)}\frac{X(g)}{g}\right]^{1/2}\times\exp\int_{y_0}^{g}\left[\frac{3-\gamma(x)/\nu(x)}{2W(x)}dx\right]. \quad (5.1)$$

(ii) The susceptibility [from Eq. (3.30)]

$$\chi^-(g)=\chi^+(g)/S(g)\ . \quad (5.2)$$

(iii) The specific heat [from Eqs. (3.31) and (3.32)],

$$C^-(g)=C^+(g)+\frac{y_0Z_1(y_0)}{8\pi g_0[Z_2(y_0)]^2}\tilde{F}(g)\times\exp\int_{y_0}^{g}\frac{3-2/\nu(x)}{W(x)}dx \quad (5.3)$$

(iv) The temperaturelike scaling field [from Eqs. (2.8), (2.16), and (2.17)],

TABLE VI. Estimates of the universal leading amplitude combinations from this work and ϵ expansion (Refs. 21, 31, and 32), high-temperature expansion (Ref. 32), and zeros Monte Carlo (Ref. 37). For experimental values, see Ref. 13.

		A^+/A^-	Γ^+/Γ^-	$R_C^+=A^+\Gamma^+/B^2$
This work		0.541±0.014	4.77±0.30	0.0594±0.0011
ϵ expansion	(ϵ)	0.54		0.066
	(ϵ^2)	0.38, 0.44, 0.17	4.9	
High temperature		0.51	5.07	0.059
Zeroes Monte Carlo		0.45±0.07		

$$t_-(g)=t_+(g)-g_0^2\frac{Z_2(y_0)Z_3(y_0)}{[y_0Z_1(y_0)]^2}\left[\exp\int_{y_0}^{g}\frac{dx}{\nu(x)W(x)}\right]\left[X(g)-X(g^*)\exp\int_{g}^{g^*}\frac{dx}{\nu(x)W(x)}\right]. \tag{5.4}$$

The value of y_0 is chosen small enough to estimate $Z_i(y_0)$ from the simple sum of the series.

These expressions allow the same numerical study of the dependence on g as that done in I. The critical singularities are all contained inside expressions already considered in I. Only the series $X(g)$, $S(g)$, and $\widetilde{F}(g)$ are new. Since the resummation method used in Sec. IV does not depend on g, we may follow the same method as in I for $y_0 \le g \le g^*$.

B. Discretized variation in g

The critical behaviors of M_0^s, χ^-, and C^- are expressed by Eqs. (5.1)—(5.4) as implicit functions of t_- via the dummy variable g. Since we cannot explicitly eliminate the g dependence to the benefit of t, we shall numerically study the functions of Eqs. (5.1)—(5.4) at discrete values g_p of g between y_0 and g^*. As already mentioned, y_0 is chosen to be very small in order to estimate the prefactors $Z_i(y_0)$ directly from their perturbative expressions. The set g_p is chosen to be the same as in I so that results for $\chi^+(g)$, $C^+(g)$, and $t_+(g)$ may be used again. This numerical study is made for the two bounds $g^*_{\max}$ and $g^*_{\min}$ of the fixed point g^* and the error in the resummation of the series in Eqs. (5.1)—(5.4) are arranged to obtain the envelope of the critical behavior. This procedure will allow the largest error estimates on the correction to scaling to be accounted for. However, the leading critical amplitudes corresponding to the envelope do not always coincide with the largest error estimates given in Table VI. This explains why the apparent errors on the universal combinations of leading critical amplitudes obtained from Table VII are underestimated in the final form of the nonasymptotic critical behaviors.

TABLE VII. Numerical values of the parameters found by adjustment of the function $G^*(|t^*|)$ of Eq. (5.5) to the primary discretized evolutions of χ^-, C^-, and M_0^s for Ising-type systems. Two sets of values are given (two successive lines) according to the bound max (upper line) and min defined in the text. These bounds give an indication of the accuracy of the work. The functions so defined reproduce the ϕ^4 model at $d=3$ for $|t^*_-|\le 2\times10^{-3}$ the universal amplitude combinations may easily be obtained and compared with Table VI (see text). In order to facilitate the comparison with Table VIII, we give the values (a_G^-) of the first correction amplitudes as defined by Eq. (5.6). The values of the correction to scaling exponent Δ are $\Delta_{\max}=0.491\,25$ and $\Delta_{\min}=0.500\,31$.

G^*	M_0^*	C_0^*	χ^*
e	0.325 16	0.108 496	1.241 94
	0.323 57	0.112 636	1.239 49
χ_1^-	9.2317×10^{-1}	3.152 14	5.4922×10^{-1}
	$8.960\,85\times10^{-1}$	3.018 23	5.7055×10^{-1}
Y_2	369.337	106.168	82.3859
	138.187	243.300	100.989
Y_3	$-2.078\,25\times10^{-1}$	$8.549\,15\times10^{-1}$	$-2.680\,11$
	$3.455\,88\times10^{-1}$	$-1.445\,36\times10^{-1}$	$-2.777\,62$
Y_4	20.5399	$-1.859\,97$	73.9688
	10.4314	49.3360	87.6054
Y_5	26.6251	8.102 49	4.992 73
	$5.386\,49\times10^{-1}$	$2.014\,43\times10^{-1}$	4.833 34
Y_6	20.5399	$-1.005\,45$	69.4425
	$-3.677\,01\times10^{-1}$	$8.706\,96\times10^{-1}$	77.2763
Y_7	-26.3094	-16.8457	$-1.744\,07$
	2.793 64	19.1640	$-1.516\,37$
X_6		$-3.880\,14$	
		$-3.575\,68$	
a_G^-	5.7169	10.9436	27.3904
	9.3672	6.5887	25.7382

Figure 4 illustrates the discretized evolutions of C^+ and C^- in terms of $|t|$ as obtained from the numerical study. For large t, one may observe the so called classical gap of the specific heat (C^+/g_0 goes to zero while C^-/g_0 goes to 3). However, as recalled below, the only part of interest is $|t|\to 0$ and the classical gap of C, as it appears in this figure, has no quantitative significance (see Sec. VI).

C. Final form of nonasymptotic critical behaviors

In order to allow easy comparison with experiments, we make, following I, a continuous interpolation of the discretized evolution obtained above by using a function

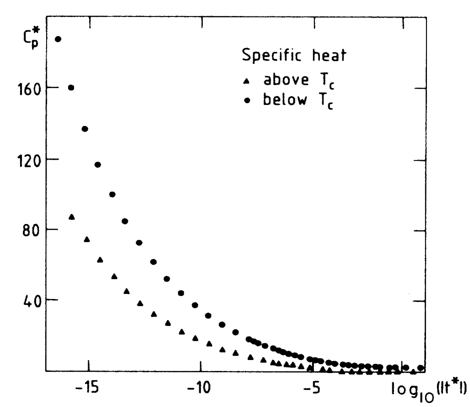

FIG. 4. Discretized evolution C_p^* of $C^*(g)$ in terms of $|t^*|$ in the two cases: $T>T_c$ (triangles) and $T<T_c$ (dots). This figure displays a crossing over from the critical behavior (near T_c) and "classical" (noncritical) behavior (far from T_c) as it comes from the renormalized ϕ^4 model. In particular one observes, far from T_c, the classical gap of the specific heat between the two phases (0.0 above T_c to $+3.0$ below T_c). This crossing over, as well as this gap, is not quantitatively correct (see text). Only at some distance to T_c the evolution begins to be quantitatively correct.

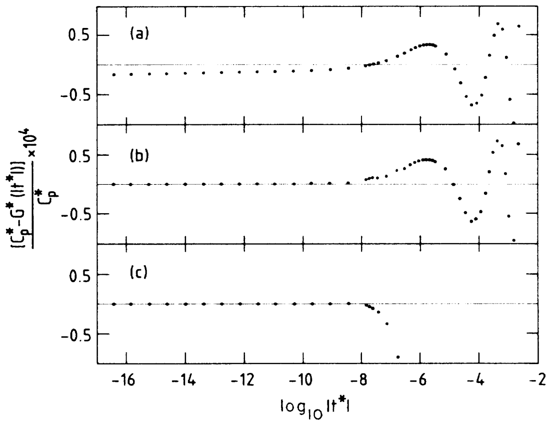

FIG. 5. Relative deviations of the function $C^*(|t^*|)$ [Eq. (5.5) and Table VII] from the discretized evolution C_p^* of the specific heat below T_c of Fig. 3. The adjustment is made up to $t^*=2\times10^{-3}$. The largest deviation is smaller than 10^{-4}. (a) The values of the parameters X_1^-, Y_i $(i=2-7)$, e, Δ, and X_6 are those of Table VII. The slight deviation in the asymptotic region is due to the rounding of the values found in the true adjustment. (b) Deviations of the true adjustment. (c) Deviations of the expansion up to the first correction to scaling of the function C^*. One sees that the oscillating tail is outside the preasymptotic critical domain D_{preas}.

of $|t|$ having the following form:

$$G^*(t^*)=X_1^-\,|t_-^*|^{-e}\prod_{k=1}^{3}(1+Y_{2k}\,|t_-^*|^{\Delta})^{Y_{2k+1}}+X_6\ . \tag{5.5}$$

This function $G^*(t^*)$ is similar to the function $F^*(t^*)$ of I; it stands for any of the dimensionless functions $M_0^s/g_0^{1/2}$, $\chi^- g_0^2$, or $C^- g_0$; e is the corresponding critical exponent (β, γ, or α), and Δ the first correction to scaling exponent. The parameters X_1^-, Y_i $(i=2-7)$, and X_6 are adjusted to reproduce the discretized evolution with a relative error of less than 10^{-4} (see Fig. 5). They are displayed in Table VII. X_6 is only needed for the specific heat and must be the same above and below T_c. It must have the same value as in I. In order to test the numerical accuracy, we have estimated it again. This explains the slight difference that is observed between this work and I in the determination of X_6.

As stressed in I, the ϕ^4 model is not adapted to reproduce real critical behaviors beyond the first nonanalytic correction to scaling. We have thus limited the interpolation of points by Eq. (5.5) to values of $|t|$ smaller than 2×10^{-3}. This range is large enough to involve the influence domain of at least three nonanalytic correction terms, as shown in Fig. 5. In addition, the work done in Ref. 11 shows that this range of $|t|$ will correspond to a range of temperature in fluids such that $|\tau| = |T-T_c|/T_c$ is less than unity. This is sufficient to allow concrete comparison with any significant experiment made near the critical point (see Sec. VI).

D. Universal ratios of the first confluent corrections to scaling

The first confluent corrections to scaling may be obtained by expanding Eq. (5.5) around $|t|=0$. Estimates of correction amplitudes are thus given by the expression

$$a_G^- = \sum_{k=1}^{3} Y_{2k}Y_{2k+1}\ , \tag{5.6}$$

in which G stands for M_0^s, χ^-, or C^- and the values of Y_i $(i=2-7)$ are taken from Table VII. From these estimates and those of I, we get values of the various universal ratios displayed in Table VIII.

Let us note that these estimates are the most precise of those obtained up to now. The previous[20,21] knowledge on these universal ratios came from ϵ-expansion framework up to ϵ^2. The ϵ expansion is, as the perturbative expansion at $d=3$, divergent[35] and necessitates the use of a resummation method such as the one used in this work. As shown by Le Guillou and Zinn-Justin,[27] a number of orders equal to at least five is needed to get estimates as precise as those of the present work. Since many experimental analyses have been made using estimates obtained from the ϵ expansion, it seems to us worthwhile to stress the lack of accuracy of these previous results. In Appendix C, we give ϵ expansions of correction amplitude ratios

TABLE VIII. Universal correction amplitude ratios and $R_{B_{cr}}^-$ obtained from Table VII (This work). A comparison is proposed with calculations $O(\epsilon^2)$ by Chang and Houghton (CH) (Ref. 20) and Nicoll and Albright (NA) (Ref. 21), for which we present the three Padé approximants (see text and Appendix C).

			a_C^+/a_C^-	a_χ^+/a_χ^-	a_M^-/a_χ^+	$R_{B_{cr}}^-$
This work			0.96±0.25	0.315±0.013	0.90±0.21	−1.334±0.044
Leading order in ϵ						−2.8
$O(\epsilon^2)$		(2,0)	−1.02	0.85	6.55	
	CH	(1,1)	1.0	0.21	1.16	
		(0,2)	0.12	0.23	−0.87	
		(2,0)	−15.42	1.46	0.27	
	NA	(1,1)	2.54	0.32	0.23	
		(0,2)	0.18	0.32	0.50	

TABLE IX. Comparison between theoretical and experimental estimates of some universal correction amplitude ratios. Our values (This work) are obtained from Table III of Ref. 10 and Table VII of this paper (not from Table VIII which would have needlessly increased the errors). The collected experimental values greatly depend on the way the data have been fitted (see Sec. VI). SF_6 (Refs. 38); Ar (Ref. 39); CO_2 (Ref. 6); He_3 (Ref. 40); Xe (Refs. 38 and 41); 3-methylpentane + nitroethane (NM) (Ref. 42); triethylamine + water (TW) (Ref. 43); triethylamine + heavy water (TD) (Ref. 44).

	a_C^-/a_C^+	a_C^+/a_M^-	a_M^-/a_χ^+	a_M^-/a_χ^-
This work	1.12 ± 0.29	1.10 ± 0.25	0.90 ± 0.21	0.29 ± 0.08
Simple fluids				
SF_6			0.7	
Ar			0.15 ± 0.04	
			0.12 ± 0.04	
CO_2	1.40			
He_3			0.41 ± 0.02	
Xe			1.4	1.5
Binary mixtures				
N-M	$0.74-1.5$			
T-W	1.	1.85 ± 0.1	0.47 ± 0.07	
T-D	-1.9			

from the work of Chang and Houghton[20] (CH) and Nicoll and Albright[21] (NA). These two sets of calculations lead to different results. If either of them is correct, it shows that the series are badly convergent and considering their shortness, we can only perform Padé approximants to get numbers. The dispersion of these numbers yields some indication on the error. The results of this Padé analysis are displayed in Table VIII. From the set of numbers so obtained, it is difficult to choose a value as the best estimate for any of the universal ratios, apart from perhaps a_χ^+/a_χ^- which, in the two papers, have values close to each other which are not very far from our result (the NA result is the closest). Let us note, however, that estimates used previously in experimental analysis were Padé (1,1) of CH which is compatible (apart from a_χ^+/a_χ^-) with the present work, but we do not know whether CH obtained the correct ϵ expansion (see Appendix C).

In Table IX, we also present a comparison between our estimates of the correction amplitude ratios and the results available from experimental analysis. There is not a clear coherence between the various experimental estimates. It is thus difficult to make a significant comparison with theory. In our opinion, as is explicitly shown in Refs. 10—12, the usual experimental analysis is not well adapted to the determination of critical behavior as long as gravitational effects have not been reduced. In general, measurements of the pure scaling behavior cannot be made without correction due to the gravity. These corrections depend essentially on the values chosen for the critical exponents and on the determination of T_c. The number of adjustable parameters is too large and the range of temperature corresponding to the pure scaling behavior is too small to allow a precise determination of the asymptotic critical behavior from present experiments. If a wider temperature domain is analyzed (in a region where the gravitational effects may be neglected), corrections to scaling must be introduced. This necessitates the introduction of a too large number of new adjustable parameters. In addition, the question of the convergence of the Wegner expansion[7] around T_c has been raised and not at all controlled. In the following section, we show, following I, how this work may be used to analyze experiments near the critical point in a more satisfactory way than has been followed to date.

VI. USE OF THE RESULTS FOR A COMPARISON WITH EXPERIMENTS

A. Adjustable parameters and test of the theory

For a concrete comparison with experiments, we must define the way that adjustable parameters enter in the nonasymptotic critical behavior obtained in the preceding section (Table VII). These free parameters must be separated into two parts corresponding to the two kinds of renormalization considered in this work (the 4-d and 3-d renormalizations). Indeed, as long as we limit ourselves to the ϕ^4 model, the only arbitrariness is contained in the renormalization. From this, we make the following remarks.

(i) Since the renormalizations are the same in going from above to below T_c, the set of adjustable parameters is unique for the two phases.

(ii) As the 3-d renormalization is not related to the critical singularities, the adjustable parameters associated to it (regular part, $\mathscr{F}_{\rm reg}$, of the free energy and T_c) should be determined independently of the critical behavior itself. Once $\mathscr{F}_{\rm reg}$ and T_c are determined, no adjustable parameters remain except those related to the 4-d renormalization. However it is often necessary to let T_c also be free in the curve fitting method used to analyze experimental data near the critical point. As for $\mathscr{F}_{\rm reg}$ it must be determined independently of the critical behavior itself.

(iii) The set of adjustable parameters associated to the 4-d renormalization is given in I (see also Ref. 45) and is limited to three. They were noted θ, ψ, and u_0. They are defined as follows:

$$|t|/g_0^2=\theta|\tau| \ , \tag{6.1a}$$

$$h/g_0^{5/2}=\psi H \ , \tag{6.1b}$$

$$g_0=u_0/l_0 \ . \tag{6.1c}$$

h is the scaling field coupled to ϕ_0 in the Hamiltonian, Eq. (1.1). The experimental quantities τ, H, and l_0 are defined, in the case of the study of the liquid-gas transition,[11] as follows:

$$\tau=(T-T_c)/T_c \ , \tag{6.2a}$$

$$H=\frac{\rho_c}{P_c}(\mu-\mu_c)\equiv\Delta\mu^* \ , \tag{6.2b}$$

$$l_0=[k_B T_c/P_c]^{1/3} \ , \tag{6.2c}$$

in which μ is the chemical potential, μ_c its critical value, ρ_c the critical density, and P_c the critical pressure of the fluid considered; k_B is the Boltzmann constant. Of course Eqs. (6.1) must be seen as the first order of a Taylor expansion around the critical point, higher terms will generate analytic confluent corrections to scaling which are considered in the following subsection.

Let us consider in detail the introduction of the adjustable parameters inside the theoretical functions displayed in Table III of I and Table VII of the present work. Let us first provide the precise notations. We shall denote by $F_{\rm th}(t,M)$ the bare theoretical free energy corresponding to $\Gamma_0^{(0,0)}(g_0,r'_0,M_0)$ considered in this work. We shall not distinguish between the two phases by a subscript $+$ or $-$ since the introduction of the free parameters is independent of the phase considered. The dimensionless theoretical functions to be compared to experiments will be noted $\chi^*_{\rm th}(t)$, $\xi^*_{\rm th}(t)$, $C^*_{\rm th}(t)$, and $M^*_{\rm th}(t)$ corresponding respectively to χg_0^2, ξg_0, $C g_0$, and $M_0^s/g_0^{1/2}$ considered in this paper and in I. The measured quantities will be noted $\xi_{\rm expt}(\tau)$, $\chi_{\rm expt}(\tau)$, $C_{\rm expt}(\tau)$, and $M_{\rm expt}(\tau)$ while the physical free energy by unit volume will be written $f(\tau,\Delta\rho^*)$ with $\Delta\rho^*=(\rho-\rho_c)/\rho_c$.

The first basic relation between theory and real systems is given by the following equality:

$$f(\tau,\Delta\rho^*)=k_B T_c F_{\rm th}(t,M)+f_{\rm reg}(\tau) \ . \tag{6.3}$$

In writing T_c instead of T in this equation, we neglect analytic corrections as in Eqs. (6.1). The term $f_{\rm reg}(\tau)$ corresponds to the regular part $\mathcal{F}_{\rm reg}$ of the free energy by unit volume which does not belong to the critical part of f.

The second fundamental relation is given by consideration of the thermodynamically conjugated variables h and M and those of μ and ρ. From the respective definitions of Legendre transforms which relate two thermodynamic potentials, we must identify

$$k_B T_c h M=(\mu-\mu_c)(\rho-\rho_c) \ . \tag{6.4}$$

Replacing the couple μ and ρ by the couple $\mu-\mu_c$, $\rho-\rho_c$ must be seen as the definition of conjugated variables associated to the singular part of the thermodynamic potentials. At $\mu=\mu_c$ and $\rho=\rho_c$ these singular parts vanish.

From Eqs. (6.1a), (6.1b), (6.2b), and (6.4) we find the first relation of practical use to compare with experiments,

$$\Delta\rho^*=u_0^3\psi M^*_{\rm th}(\theta|\tau|) \ . \tag{6.5}$$

The relation between $\chi_{\rm expt}$ and $\chi_{\rm th}$ follows now from their respective definitions,

$$1/\chi_{\rm expt}=\left[\frac{\partial^2 f}{\partial\rho^2}\right]_T \ , \tag{6.6a}$$

$$1/\chi_{\rm th}=\left[\frac{\partial^2\mathcal{F}_{\rm th}}{\partial M^2}\right]_t \ . \tag{6.6b}$$

We thus obtain

$$\chi^*_{\rm expt}(\tau)=u_0^3\psi^2\chi^*_{\rm th}(\theta|\tau|) \ , \tag{6.7}$$

in which $\chi^*_{\rm expt}$ is dimensionless,[23]

$$\chi^*_{\rm expt}(\tau)=\frac{P_c}{\rho_c^2}\chi_{\rm expt} \ . \tag{6.8}$$

Similarly, we can check the following relation for the specific heat:

$$C^*_{\rm expt}(\tau)=u_0^3\theta^2 C^*_{\rm th}(\theta|\tau|)+B_{\rm bg}(\tau) \ , \tag{6.9}$$

with

$$C^*_{\rm expt}(\tau)=\frac{l_0^3}{k_B}C_{\rm expt}(\tau) \tag{6.10}$$

and $B_{\rm bg}(\tau)$ is related to the second derivative with respect to τ of $f_{\rm reg}(\tau)$. Let us note that $C_{\rm expt}(\tau)$ represents, here, the specific heat per unit volume.

Finally, we also have

$$\xi^*_{\rm expt}(\tau)=\xi^*_{\rm th}(\theta|\tau|)/u_0 \ , \tag{6.11}$$

with

$$\xi^*_{\rm expt}(\tau)=\xi_{\rm expt}(\tau)/l_0 \ . \tag{6.12}$$

The set of Eqs. (6.5), (6.7), (6.9), and (6.10) provides the practical relations between the functions of t^* calculated in this work and in I with the usual[23] dimensionless experimental quantities.

There are only three adjustable critical parameters u_0, θ, and ψ (four if we also let T_c free) to describe the critical behavior of six measurable quantities ($\chi^\pm$, $C^\pm$, ξ^+, and M) in a range of temperature not restricted to the close vicinity of T_c. The test of coherence between theory and experiment is the determination of unique values for u_0, θ, and ψ for the six different measurements. Notice that this corresponds to a check of all the theoretical estimates of universal amplitude combinations (including the first confluent correction of the present work and of Ref. 36) simultaneously with that of the theoretical estimates of exponents.[4] As explained in I, another advantage of the theoretical functions proposed is that they also provide the possibility of determining their own domain of validity by testing the influence of the correction to scaling higher than the first. It is worthwhile, at this stage, to recall how this can be done. We shall illustrate this with the specific-heat case after having pointed out its very peculiar and interesting form.

B. Specific-heat case and corrections to scaling

Let us first emphasize the particularity of the critical singularity of $C^{\pm}$. As already mentioned,[18] it is not, as in the case of $\chi^{\pm}$ or M_0^s, a pure scaling form. It also involves a critical constant, called $B_{\rm cr}$. This critical constant is related to the term X_6 in Eq. (5.5) whose value has been determined by the interpolation of Sec. V C (see Table VII of the present work and Table III of I). Let us insist on the fact that X_6 has not been arbitrarily chosen in the critical behavior of $C^{\pm}$. In particular it is independent of the constant of integration $C^{+}(y_0)$ which appears in Eq. (4.21) of I. This latter constant has been estimated directly from the series $C^{+}(g)$ for $g=y_0$. Since y_0 has been chosen to be very small (say $y_0 \simeq 0.05$), the integration constant cannot interfere with the critical behavior which corresponds to values of g near g^*, which is about 30 times larger than y_0. The constant X_6 is thus entirely determined by the interpolation process.

Table III of I shows a very interesting correlation between X_1^+ (noted simply X_1 in I), α and X_6. For $\alpha>0$ ($n=1$), X_6 is negative while X_1^+ is positive and about twice as small as X_6. For $\alpha<0$ ($n=3$), X_6 is positive while X_1^+ is negative and about twice as large as X_6. In the case $\alpha=0$, X_1^+ and X_6 are (very likely) infinite with $X_1^+/X_6=-1$. This is well indicated by the case $n=2$ of Table III in which $\alpha \simeq 0$ and $X_1^+ \simeq -X_6$ with large values (the smaller the value of α the larger the values of X_1^+ and X_6). The case $\alpha=0$ was expected[7] to be so in order to allow a smooth continuity at $\alpha=0$ which corresponds to a logarithmic critical behavior of $C_{\rm sing}$. In that case we usually write $C_{\rm sing} \simeq (A^+/\alpha)(\tau^{-\alpha}-1)$ which includes the critical constant $B_{\rm cr}=-A^+/\alpha$. To our knowledge the work done in I is the first which quantitatively exhibits this special relation between $B_{\rm cr}$ and A^+ for small values of α. However, it also shows that the above expression for $C_{\rm sing}$ is only valid for infinitesimal values of α contrary to what is very often written.

At $\alpha=0$ we thus have a universal relation between A^+ and $B_{\rm cr}$, namely, $A^+/(\alpha B_{\rm cr})=-1$. From the same argument of continuity at $\alpha=0$ already used above, we may expect that a universal combination again relates A^+ to $B_{\rm cr}$ for $\alpha \neq 0$. This relation may, of course, be modified by the fact that α is no longer equal to zero. It has been shown[18,19] that the true universal combination is $R_{B_{\rm cr}}^+ = A^+ |a_C^+|^{\alpha/\Delta}/(\alpha B_{\rm cr})$ in which a_C^+ is the amplitude of the first nonanalytic correction to scaling. In the limit $\alpha \to 0$, $R_{B_{\rm cr}}^+$ reduces to $A^+/(\alpha B_{\rm cr})$ as expected. Only explicit calculations (except for $\alpha=0$) may give the value of $R_{B_{\rm cr}}^+$. Results at leading order in ϵ for $R_{B_{\rm cr}}^{\pm}$ may be found in Ref. 19. Estimates for $n=1$, 2, and 3 may be found from I (see also Ref. 19). Owing to the universal ratios A^+/A^- and a_C^+/a_C^- the universal combination $R_{B_{\rm cr}}^-$ exists whose value for Ising-type systems ($n=1$), estimated directly at $d=3$, is displayed in Table VIII of the present paper.

In the following we shall discuss the comparison with experiments in the case of C. It must be kept in mind that the singular part of C will always involve the critical constant $B_{\rm cr}$ through X_6 term in $C_{\rm th}$.

Equations (6.9) and (6.10) provide the link between the theoretical functions $C_{\rm th}^*$ (taken from I for $\tau>0$ or from this work for $\tau<0$) and measured specific heat per unit volume $C_{\rm expt}$. Figure 4 displays the relative behaviors of $C_{\rm th}^*$ in the two phases. We see that, when $t \to \pm\infty$, the two functions have different limits. In the case $t>0$, $C_{\rm th}^* \to 0$, while for $t<0$, $C_{\rm th}^* \to 3$. This is an illustration of the gap of C predicted by Landau theory. It is clear that such behaviors for large t cannot describe the true behaviors of $C_{\rm expt}$ far from T_c. In fact, as already mentioned, the present study does not account for corrections to scaling induced by higher transients. These effects, which convey in critical behavior the necessary nonzero atomic spacing,[31] modify in an essential way the crossover function from criticality to noncriticality obtained by considering only a pure ϕ^4 Hamiltonian.[46,47] We question whether this means that one must consider higher interactions in the theory to reproduce real critical behavior, and determine that it does down to a certain distance to T_c at which one may neglect these higher transients provided that one sets up the renormalization process.[31] The effects of higher transients within the critical behavior is twofold.[8]

(1) They induce higher corrections to scaling controlled by exponents different from Δ.[7] These correction terms are not reproduced in the theoretical function of t obtained in this work and in I. However, they are dominated, in the vicinity of T_c, by the first correction controlled by Δ. Nevertheless it can be noticed that the exponent Δ_2 of the first higher nonanalytic correction has been estimated[48] to be of order 2Δ for Ising-type systems. In other words, the second nonanalytic correction to scaling of the ϕ^4 model may also be considered as reproducing the temperature dependence of the true second nonanalytic correction to scaling; only the amplitude could be incorrect.[47] The control of the convergence of the Wegner expansion that we get from the theoretical function $C_{\rm th}^*(t)$ provides the opportunity of estimating the width of the preasymptotic critical domain in which only the first correction to scaling is relevant (see I and Refs. 11 and 12).

(2) They contribute to the amplitudes of the pure scaling and first correction terms. This effect is taken into account by the renormalization transformations provided that a finite cutoff is kept in the RG analysis.[18,19,31] These effects are conveyed, in the function $C_{\rm th}^*$, by the adjustable parameters θ, ψ, and u_0. This particularity of critical behavior is very often forgotten. For example, the idea that the universality could be entirely studied from the consideration of the single ϕ^4 model without considering any renormalization process, is very often used but it is incorrect.[31] It has been shown[49] that the true continuous limit of the Ising model involves an *infinite* set of couplings. Only this infinite set may be reduced to one (the ϕ^4 coupling) after considering Wilson's renormalization transformations. Effect (1) above (corrections to scaling) of higher transients may be progressively neglected (provided that the distance to T_c is sufficiently small) but effect (2) modifies the critical amplitudes that should have been obtained from the study of the pure ϕ^4 model without renormalization. The nonconsideration of higher

transients is unseparable from the consideration of the renormalization process (see Ref. 31).

Higher transient effects have an important consequence on the specific heat critical behavior. It is the universal combination $R^{\pm}_{B_{\rm cr}}$ which relates the leading ($A^{\pm}$) and first correct ($a_c^{\pm}$) amplitudes to the critical constant $B_{\rm cr}$ [related to X_6 of Eq. (5.5)]. This universal combination cannot be derived from the ϕ^4 model without considering the renormalization process.[18,19,31] In Ref. 19, $R^{\pm}_{B_{\rm cr}}$ is calculated at leading order in ϵ and the close link between its universality and the renormalization is explicitly shown. In Table VIII we provide an estimate at $d=3$ of $R^{-}_{B_{\rm cr}}$ obtained from the present work.

The universal quantity $R^{\pm}_{B_{\rm cr}}$ is important for the comparison with experiments since it concerns only the specific heat. One must first determine the noncritical regular part $B_{\rm bg}(\tau)$, related to $\mathscr{F}_{\rm reg}$ set equal to zero to obtain the critical behavior. In I we proposed a determination of $B_{\rm bg}$ from an interpolation in the critical domain of the regular behavior of $C_{\rm expt}$ far from T_c. However, this does not correspond to the true definition of $B_{\rm bg}$ since it assumes that the critical degrees of freedom no longer fluctuate outside the critical domain. This assumption is obviously incorrect. Nevertheless, notice that in systems where α is small, one expects a large value of $B_{\rm cr}$ with respect to $B_{\rm bg}$. For example, in Ref. 12, we found that $B_{\rm bg}$, defined as in I, was negligible considering the accuracy in the determination of $B_{\rm cr}$. For systems with larger values of α, one must check *a posteriori* the relative importance of $B_{\rm bg}$ and $B_{\rm cr}$. It is worthwhile to note that the determination proposed in I gives an upper bound on $B_{\rm bg}$.

The main consequence of $R^{\pm}_{B_{\rm cr}}$ is the possible presence of large analytic corrections to scaling. In order to reestablish them inside the theoretical function $C_{\rm th}$, we must consider higher orders in Eqs. (6.1). These equations are valid in the close vicinity of T_c and are a consequence of Wilson's hypothesis of analyticity for the various couplings in the Hamiltonian. Further away from T_c, they must include higher terms which will mix τ and H. In the following, we shall only consider higher terms in τ which are responsible for confluent analytic corrections to scaling along the critical isochore. Those corresponding to H lead to another kind of corrections in the critical behavior of C which bring the critical exponent γ into play[50] but they are smaller than the true confluent analytic correction to scaling.

In order to account for these latter terms in a correct way, we must no longer restrict T to its critical value T_c in Eqs. (6.3) and in the definition of $C_{\rm expt}$,

$$C_{\rm expt}=-T\frac{d^2f}{dT^2} \ . \tag{6.13}$$

We introduce three new adjustable parameters θ_1, ψ_1, and u_1 defined[12] by modifying Eqs. (6.1)

$$|t|/g_0^2=\theta|\tau|(1+\theta_1|\tau|)\ , \tag{6.14a}$$

$$h/g_0^{5/2}=\psi H(1+\psi_1|\tau|)\ , \tag{6.14b}$$

$$g_0=\frac{u_0}{l_0}(1+u_1|\tau|)\ . \tag{6.14c}$$

For the specific-heat case, only θ_1 and u_1 (noted θ_1 and g_{01} in Ref. 12) will contribute through corrections to Eq. (6.9), which in the preasymptotic domain, will now read

$$C^*_{\rm expt}(\tau)=u_0^3\theta^2[C^*_{\rm th}(\theta|\tau|)+X_1^{\pm}\theta^{-\alpha}A_1|\tau|^{1-\alpha}+X_6A_2|\tau|]+B_{\rm bg}(\tau)\ , \tag{6.15}$$

in which A_1 and A_2 are functions of θ_1 and u_1 and are easy to obtain,

$$A_1=[4-2\alpha+3(3-\alpha)u_1+(2-\alpha)(3-\alpha)\theta_1]/(1-\alpha)\ , \tag{6.16a}$$

$$A_2=4+9u_1+6\theta_1\ . \tag{6.16b}$$

The range of τ for which Eq. (6.15) is defined is, of course, smaller than that for Eq. (6.9). This is because of the limited validity of Eqs. (6.14). Before using Eq. (6.15), we must try using Eq. (6.9) in order to estimate the relative influence of the first, second, etc. corrections corresponding to the available range of τ (see I). Only then may the analytic corrections be introduced as in Eq. (6.15) for values of τ smaller than those at which second corrections becomes to be important. A test of validity of the theory will be a reproduction of experimental data for reasonable values of u_1 and θ_1. We can state the following about these two new parameters.

Equations (6.16) clearly show that, whatever the values of u_1 and θ_1, the coefficients A_1 and A_2 are close to each other since α is small ($\alpha\simeq0.11$). Hence, the same remarks as in Ref. 12 may be reproduced here for Ising-type systems. From the definitions of u_1 and θ_1 [Eqs. (6.14)], we reasonably expect that they are of order unity; in that case, we have $A_1\simeq20$ and $A_2\simeq19$.

Comparison between correction proportional to A_1 in Eq. (6.15) and first nonanalytic correction inside $C^*_{\rm th}$ is like the comparison between $A_1\tau$ to $a_c^{\pm}|\theta\tau|^{\Delta}$ (a_c^- is given in Table VII and a_c^+ may be obtained from I, one has $a_c^-\simeq8.5$ and $a_c^+\simeq7.8$). Now θ, for liquid-gas systems is[11] of order 10^{-2}. Hence the two corrections are, respectively, 20τ and $5.5|\tau|^{0.5}$. The amplitude of the analytic correction is only four times as large as that of the nonanalytic correction. This is a very different situation than that for ^{4}He for which a_c^+ is about 50 times smaller. We may thus conclude that the analytic confluent corrections will not have an important as large (50 times less important) as in ^{4}He within $D_{\rm preas}$.

Let us note however that they cannot be completely neglected as far as more than one correction is needed as, for example, in Ref. 38.

VII. CONCLUSION

As already mentioned above, experimentalists were bothered with too large a number of adjustable parameters in reproducing their data. The functions that we propose, not only involve a small number of physically justified adjustable parameters, but reproduce with a great accuracy real critical behavior in $D_{\rm preas}$. They contain all the universal characteristics qualitatively known but which, up to now, have not all been quantitatively determined. We describe the way that experimentalists must use the functions to analyze their data even in the case where analytic corrections are not negligible.

Together with I, this paper provides a rather complete view (at least for Ising-type systems) of the best theoretical knowledge (up to now) on second-order phase transitions. They provide experimentalists with a unified (the set of adjustable parameters is the same in the two phases) version of what the renormalization-group theory can say on their systems.

ACKNOWLEDGMENTS

Two of the authors (C. Bagnuls and C. Bervillier) thank Professor J. Reeve for having kindly collected and communicated values of Feynman integrals in three dimensions for pure ϕ^3 theory.

APPENDIX A: PRACTICAL USE OF 3-d RENORMALIZATION ON FEYNMAN INTEGRALS. EXAMPLES

The object of this appendix is to give the practical definition of 3-d renormalization used in this work and in I. As explained in the text, it corresponds to the following: (1) A mass shift

$$r_0 = r_0' + \delta r_0(\epsilon) , \tag{A1}$$

in which $\delta r_0(\epsilon)$ subtracts the poles at $\epsilon=1$ primarily contained in $\Gamma_0^{(0,2)}$. (2) Adding constants to the free energy and its first derivative with respect to r_0' in order to subtract the poles at $\epsilon=1$ contained in $\Gamma_0^{(0,0)}$ and $\Gamma_0^{(1,0)}$ which have not been eliminated by the mass shift.

The relation of the arbitrariness introduced by such subtractions to the nonuniversality of T_c and of the regular part $\mathscr{F}_{\text{reg}}$ of the free energy is discussed in the text (see Sec. III). Here, we are interested only in the technical aspect of subtracting the poles at $\epsilon=1$.

As explained in I, $\delta r_0(\epsilon)$ may be defined as subtracting exactly the pole at $\epsilon=1$ of the Feynman integral (FI) $I_1^{(2,1)}(\epsilon)$ associated to the graph drawn in Fig. 6(a). In that case the contribution to $\Gamma_0^{(0,2)}$ of the 3-d renormalized integral of interest would be, at $d=3$, of the form,[10]

$$I_1(r_0',g_0) = -\tfrac{1}{6} g_0^2 [f_2(1)\ln(r_0'/g_0^2) + C_1] , \tag{A2}$$

in which $f_2(1)$ is the residue of the pole at $\epsilon=1$ of $I_1^{(2,1)}(\epsilon)$ and C_1 is a constant to be calculated. Due to the arbitrariness introduced by the subtraction, C_1 may be set equal to zero. In that case δr_0 no longer subtracts the pure pole. In order to calculate $f_2(1)$, let us perform the derivative, at g_0 fixed, with respect to r_0' of Eq. (A2),

$$\left.\frac{\partial I_1(r_0',g_0)}{\partial r_0'}\right|_{g_0} = -\tfrac{1}{6} g_0^2 f_2(1)/r_0' . \tag{A3}$$

This quantity is equal to the contribution to $\Gamma_0^{(1,2)}$, associated to the graph drawn in Fig. 6(b). Let us denote $J_1(1)$ this contribution,

$$J_1(1) = \tfrac{1}{2} g_0^2 h_1(1)/r_0' . \tag{A4}$$

Now $h_1(1)$ is already calculated[14] and its value is [after division by the geometrical factor A_d' of Eq. (3.2)],

$$h_1(1) = \tfrac{2}{3} . \tag{A5}$$

We thus obtain

$$f_2(1) = -2 . \tag{A6}$$

which, in Table I, yields

$$I_1^{(2,1)}(1) = -2\ln(r_0'/g_0^2) . \tag{A7}$$

This choice of $\delta r_0(\epsilon)$ allows us to deduce the value, at $d=3$, of any FI primarily divergent at $d=3$ from calculations done in Ref. 14. Let us illustrate this on the calculation of the FI $I_1^{(5,2)}$ and $I_2^{(5,2)}$ corresponding respectively to the graphs drawn in Figs. 7(a) and 7(b) and whose value is lacking in Table I.

Although explicitly considered in Ref. 14, these two integrals were not calculated at $d=3$ since they diverge. Instead, the divergent part was subtract at zero momentum according to the definition of $\delta\overline{m}^2$ by Eq. (2.4). Hence the effective calculations done in Ref. 14, concerning these two integrals, correspond to the following combinations:

$$\lim_{\epsilon\to 1} I_1^{(5,2)}(\epsilon) - U_1(\epsilon) I_1^{(2,1)}(\epsilon) \equiv N_1 , \tag{A8a}$$

$$\lim_{\epsilon\to 1} I_2^{(5,2)}(\epsilon) - U_2(\epsilon) I_1^{(2,1)}(\epsilon) \equiv N_2 , \tag{A8b}$$

where $U_1(\epsilon)$ and $U_2(\epsilon)$ are FI (finite at $d=3$) associated, respectively, to the graphs drawn in Figs. 7(c) and 7(d).

It is clear that Eqs. (A8) correspond to a choice for subtracting the pole at $\epsilon=1$ contained in the divergent part of $I_k^{(5,2)}$ ($k=1,2$). The difference, compared to our choice of δr_0 defined just above, is a constant like C_1 in Eq. (A2). This constant is equal to the following limits for $k=1,2$:

$$U_k(1)\{\lim_{\epsilon\to 1}[I_1^{(2,1)}(\epsilon) - g_0^{2/\epsilon} f_2(1)/(\epsilon-1)]\} . \tag{A9}$$

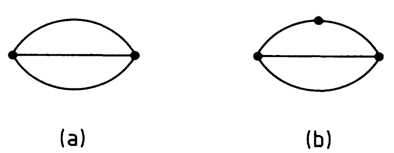

FIG. 6. (a) Graph of the Feynman integral $I_1^{(2,1)}(\epsilon)$. (b) Graph contributing to $\Gamma_0^{(1,2)}$.

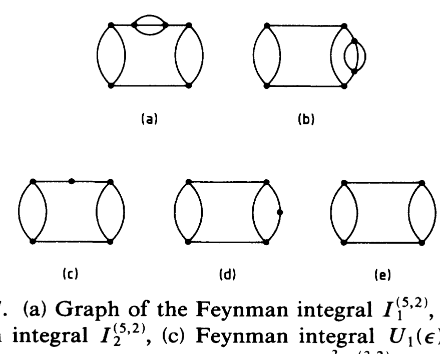

FIG. 7. (a) Graph of the Feynman integral $I_1^{(5,2)}$, (b) that for Feynman integral $I_2^{(5,2)}$, (c) Feynman integral $U_1(\epsilon)$, (d) Feynman integral $U_2(\epsilon)$, (e) Feynman integral $\frac{3}{2}I_1^{(3,2)}$.

The expression inside the curly brackets is indeed equal to the 3-d renormalized value $I_1^{(2,1)}(1)$ given in Table I [see also Eq. (A7)]. Finally we have, for $k=1,2$,

$$I_k^{(5,2)}(1)=N_k+U_k(1)I_1^{(2,1)}(1) \ . \tag{A10}$$

In this expression each term is now finite at $d=3$. However $U_k(1)$ ($k=1,2$) were not explicitly calculated in Ref. 14. Fortunately the combination $(r_0')^{-5/2}[4U_1(1)+2U_2(1)]$ is equal to minus the derivative with respect to r_0' of $(r_0')^{-3/2}I_1^{(3,2)}$ associated to the graph drawn in Fig. 7(e). We thus have

$$4U_1(1)+2U_2(1)=\tfrac{3}{2}I_1^{(3,2)} \ . \tag{A11}$$

Finally we obtain the values of the FI which were lacking in Table I via the following combination:

$$4I_1^{(5,2)}(1)+2I_2^{(5,2)}(1)=4N_1+2N_2+\tfrac{3}{2}I_1^{(3,2)}(1)I_1^{(2,1)}(1) \ . \tag{A12}$$

From the values of N_1 and N_2 given in Ref. 14: $N_1=-0.019\,892\,843\,4$ and $N_2=-0.020\,336\,438\,7$ and using Table I, we obtain

$$4I_1^{(5,2)}(1)+2I_2^{(5,2)}(1)=-0.040\,229\,281-1.558\,293\,723\ln(r_0'/g_0^2) \ . \tag{A13}$$

APPENDIX B: COMPLETE EXPANSION UP TO FIFTH-LOOP ORDER FOR $\Gamma_{0,-}^{(0,0)}$

In this appendix we give an explicit equation which provides, up to fifth-loop order, the free energy from the knowledge of its one-vertex irreducible part. Vertex reducibility corresponds to the possibility of obtaining disconnected graphs from a connected graph by cutting one four-leg vertex into two equal parts. In the following we shall denote the irreducible and the reducible parts of the free energy respectively by $\widetilde{\Gamma}$ and $\overline{\Gamma}$. In particular, the contribution, Γ_b, to bth-loop order of the free energy [Eq. (3.1)] reads

$$\Gamma_b=\widetilde{\Gamma}_b+\overline{\Gamma}_b \ . \tag{B1}$$

Once a reducible graph has been cut into two parts, in each part so obtained the four-leg vertex cut has become a two-leg vertex (a mass insertion). Now the mass insertion is related to the derivative with respect to the mass acting on the Feynman integral associated to the graph without the two-leg vertex. We shall use this property to get the perturbative expression of the free energy from the calculation of its irreducible parts.

Let us define the following combination:

$$D_b=\sum_{k=1}^{b-1}\frac{1}{k!}(\partial\widetilde{\Gamma}_1)^k\partial^k\widetilde{\Gamma}_{b-k} \ , \tag{B2}$$

in which

$$\partial=\frac{\partial}{\partial\tilde{r}_0} \ . \tag{B3}$$

We then have

$$\overline{\Gamma}_2=\tfrac{1}{2}D_2 \ ,$$

$$\overline{\Gamma}_3=D_3 \ ,$$

$$\overline{\Gamma}_4=D_4+\partial\widetilde{\Gamma}_1\partial\widetilde{\Gamma}_2\partial^2\widetilde{\Gamma}_1+\tfrac{1}{2}(\partial\widetilde{\Gamma}_2)^2+\tfrac{1}{2}(\partial\widetilde{\Gamma}_1)^2(\partial^2\widetilde{\Gamma}_1)^2 \ ,$$

$$\begin{aligned}\overline{\Gamma}_5=D_5&+\partial\widetilde{\Gamma}_2\partial\widetilde{\Gamma}_3+\partial\widetilde{\Gamma}_1\partial^2\widetilde{\Gamma}_1\partial\widetilde{\Gamma}_3+\tfrac{1}{2}\partial^2\widetilde{\Gamma}_1(\partial\widetilde{\Gamma}_2)^2\\&+\partial\widetilde{\Gamma}_1\partial\widetilde{\Gamma}_2\partial^2\widetilde{\Gamma}_2+\tfrac{1}{2}(\partial\widetilde{\Gamma}_1)^2\partial^3\widetilde{\Gamma}_1\partial\widetilde{\Gamma}_2\\&+\partial^2\widetilde{\Gamma}_1(\partial\widetilde{\Gamma}_1)^2\partial^2\widetilde{\Gamma}_2+\partial\widetilde{\Gamma}_1(\partial^2\widetilde{\Gamma}_1)^2\partial\widetilde{\Gamma}_2\\&+\tfrac{1}{2}(\partial\widetilde{\Gamma}_1)^3\partial^2\widetilde{\Gamma}_1\partial^3\widetilde{\Gamma}_1+\tfrac{1}{2}(\partial\widetilde{\Gamma}_1)^2(\partial^2\widetilde{\Gamma}_1)^3 \ ,\end{aligned}$$

which is the desired equation. Notice that it accounts for the correct weight and sign associated to each individual contribution of the Feynman integral to Eq. (3.2).

APPENDIX C: ϵ EXPANSIONS FOR CORRECTION AMPLITUDE RATIOS

In this appendix we present the respective expansions up to ϵ^2 of Chang and Houghton[20] (CH) and of Nicoll and Albright[21] (NA) relative to universal first correction amplitude ratios. Although they proceed from the same theoretical scheme, their results differ.

$$(a_\chi^+/a_\chi^-)_{\mathrm{CH}}=2^{-\Delta}(1-\tfrac{3}{2}\epsilon+\epsilon^2[\tfrac{3727}{972}-\tfrac{16}{9}\zeta(3)])=2^{-\Delta}(1-1.5\epsilon+1.7\epsilon^2)$$

$$(a_\chi^+/a_\chi^-)_{\mathrm{NA}}=2^{-\Delta}(1-\tfrac{3}{2}\epsilon+\tfrac{277}{108}\epsilon^2)=2^{-\Delta}(1-1.5\epsilon+2.57\epsilon^2)$$

$$(a_C^+/a_C^-)_{\mathrm{CH}}=2^{-\Delta}\{1+\tfrac{3}{2}\epsilon+\epsilon^2[\lambda-\zeta(2)-3\zeta(3)+\tfrac{5}{36}]\}=2^{-\Delta}(1+1.5\epsilon-3.94\epsilon^2)$$

$$(a_C^+/a_C^-)_{\mathrm{NA}}=2^{-\Delta}\{3+\tfrac{9}{2}\epsilon+3\epsilon^2[\lambda-9\zeta(3)-\tfrac{239}{1944}]\}=2^{-\Delta}(3+4.5\epsilon-29.3\epsilon^2)$$

$$(a_M^-/a_\chi^+)_{\mathrm{CH}}=2^{\Delta}\left[1-\epsilon+\epsilon^2\left[\frac{7\lambda}{6}+\frac{\zeta(2)}{2}+2\zeta(3)+\frac{1}{27}\right]\right]=2^{\Delta}(1-\epsilon+4.63\epsilon^2)$$

$$(a_M^-/a_\chi^+)_{\mathrm{NA}}=2^{\Delta}\left[1-\epsilon-\epsilon^2\left[\frac{\lambda}{2}-\tfrac{2}{3}\zeta(3)+\tfrac{325}{13\,122}\right]\right]=2^{\Delta}(1-\epsilon+0.19\epsilon^2) \ ,$$

in which $\lambda=1.171\,854$, $\zeta(2)=1.6449$, and $\zeta(3)=1.202\,06$. The various Padé approximants corresponding to the above expansions are displayed in Table VIII.

Let us only indicate that NA's results for leading amplitude combinations are in agreement with the correct ϵ expansion for A^+/A^- and R_ξ which were incorrect[51] in Ref. 33 and in CH's work. The correct results are the following:

$$\frac{A^+}{A^-}=\frac{n}{4}2^{\alpha}\left[1+\epsilon+\epsilon^2\left[\frac{3n^2+26n+100}{2(n+8)^2}+\frac{(4-n)(n-1)}{2(n+8)^2}\zeta(2)-\frac{3(5n+22)}{(n+8)^2}\zeta(3)-\frac{9(4-n)\lambda}{2(n+8)^2}\right]\right]$$

$$(R_\xi^+)^d=\frac{n}{4}K_d\left[1+\epsilon\frac{n-1}{n+8}+\epsilon^2\left[\frac{(n+2)(3n^2+46n-4)}{4(n+8)^3}+\frac{\zeta(2)}{4}+\frac{7(n+2)\lambda}{3(n+8)^2}\right]\right],$$

in which K_d is the surface of the d-dimensional sphere divided by $(2\pi)^d$. For a recent estimate of A^+/A^- for $n>1$, see Ref. 52.

---

[1]See, for example, *Phase Transitions and Critical Phenomena*, edited by C. Domb and M. S. Green (Academic, New York, 1976), Vol. VI.

[2]G. Parisi, J. Stat. Phys. **23**, 49 (1980).

[3]G. A. Baker Jr., B. G. Nickel, M. S. Green, and D. I. Meiron, Phys. Rev. Lett. **36**, 1351 (1976); Phys. Rev. B **17**, 1365 (1978).

[4]J. C. Le Guillou and J. Zinn-Justin, Phys. Rev. B **21**, 3976 (1980).

[5]C. Bervillier and C. Godrèche, Phys. Rev. B **21**, 5427 (1980).

[6]In Proceedings of the Cargèse Summer Institute on Phase Transitions, Cargèse, 1980, edited by M. Lévy, J. C. Le Guillou and J. Zinn-Justin (Plenum, New York).

[7]F. J. Wegner, Phys. Rev. B **5**, 4529 (1972) and Ref. 1.

[8]K. G. Wilson and J. Kogut, Phys. Rep. **12C**, 77 (1974).

[9]M. R. Moldover in Ref. 1; J. M. H. Levelt-Sengers and J. V. Sengers, in *Perspectives in Statistical Physics*, edited by H. J. Ravechè (North-Holland, Amsterdam, 1981), p. 239.

[10]C. Bagnuls and C. Bervillier, Phys. Rev. B **32**, 7209 (1985).

[11]C. Bagnuls, C. Bervillier, and Y. Garrabos, J. Phys. (Paris) Lett. **45**, L127 (1984).

[12]C. Bagnuls and C. Bervillier, Phys. Lett. **112A**, 9 (1985).

[13]C. Bagnuls, C. Bervillier, and E. Boccara, Phys. Lett. **103A**, 411 (1984).

[14]B. G. Nickel, D. I. Meiron, and G. A. Baker (unpublished).

[15]D. J. Wallace in Ref. 1.

[16]E. Brézin, J. C. Le Guillou, and J. Zinn-Justin in Ref. 1.

[17]G. Ahlers, in *Phase Transitions and Critical Phenomena*, Ref. 6.

[18]C. Bagnuls and C. Bervillier, Phys. Lett. **107A**, 299 (1985).

[19]C. Bagnuls and C. Bervillier, Phys. Lett. **115A**, 84 (1986).

[20]M. C. Chang and A. Houghton, Phys. Rev. Lett. **44**, 785 (1980); Phys. Rev. B **21**, 1881 (1980).

[21]J. F. Nicoll and P. C. Albright, Phys. Rev. B **31**, 4576 (1985).

[22]B. R. Heap, J. Math. Phys. **7**, 1583 (1966).

[23]J. V. Sengers and J. M. H. Levelt Sengers, in *Progress in Liquid Physics*, edited by C. A. Croxton (Wiley, New York, 1978), p. 103.

[24]K. Symanzik, Lett. Nuovo Cimento **8**, 771 (1973).

[25]In addition to critical degrees of freedom, real systems also contain noncritical variables which will contribute to $\mathscr{F}_{\text{reg}}$. The question of experimentally determining this contribution is discussed in Sec. VI; J. C. Wheeler (private communication).

[26]S. Weinberg, Phys. Rev. D **8**, 3497 (1973).

[27]J. C. Le Guillou and J. Zinn-Justin, J. Phys. (Paris) Lett. **46**, L137 (1985)

[28]K. G. Chetyrkhin, A. L. Kataev, and F. V. Tkachov, Phys. Lett. **99B**, 147 (1981); *ibid.* **101**, 457 (1981); K. G. Chetyrkhin, S. G. Gorishny, S. A. Larin, and F. V. Tkachov, Phys. Lett. **132B**, 351 (1983); D. I. Kazakov, *ibid.* **133B**, 406 (1983); S. G. Gorishny, S. A. Larin and F. V. Tkachov, Phys. Lett. **132A**, 120 (1984).

[29]We will see below that this quantity corresponds to the singular part of the free energy per unit volume and divided by k_BT. It will also be noted sometimes in the following by $\mathscr{F}_{\text{sing}}$ or even $\mathscr{F}$ as the real free energy itself when no confusion can occur.

[30]For the sake of notational simplicity, the subscript $+$ (or $-$) is not recalled since the primitive uv divergences in which we are interested are the same above and below T_c.

[31]C. Bagnuls and C. Bervillier, Phys. Rev. B **34**, 1997 (1986).

[32]A. Aharony and P. C. Hohenberg, Phys. Rev. B **13**, 3081 (1976).

[33]C. Bervillier, Phys. Rev. B **14**, 4964 (1976).

[34]L. N. Lipatov, Zh. Eksp. Teor. Fiz. **72**, 411 (1977) [Sov. Phys.—JETP **45**, 216 (1977)].

[35]E. Brézin, J. C. Le Guillou and J. Zinn-Justin, Phys. Rev. B **15**, 1544 (1977).

[36]C. Bagnuls and C. Bervillier, Phys. Rev. B **24**, 1226 (1981).

[37]E. Marinari, Nucl. Phys. B **235**, 123 (1984).

[38]H. Guttinger and D. S. Cannell, Phys. Rev. A **24**, 3188 (1981).

[39]M. A. Anisimov, A. T. Berestov, V. P. Voronov, Y. F. Kiyachenko, B. A. Koval'chuk, V. M. Malyskev, and V. A. Smirnov, Zh. Eksp. Teor. Fiz. **76**, 1661 (1979) [Sov. Phys.—JETP **49**, 844 (1979)]; M. Chandrasekhar and P. W. Schmidt, Phys. Rev. B **25**, 6730 (1982).

[40]C. Pittman, T. Doiron, and H. Meyer, Phys. Rev. B **20**, 3678 (1979).

[41]D. Balzarini and O. G. Mouritsen, Phys. Rev. A **28**, 3515 (1983).

[42]G. Sanchez, M. Meichle, and C. W. Garland, Phys. Rev. A **28**, 1647 (1983).

[43]J. Thoen, E. Bloemen, and W. Van Dael, J. Chem. Phys. **68**, 735 (1978); A. Bourgou and D. Beysens, Phys. Rev. Lett. **47**, 257 (1981).

[44]E. Bloemen, J. Thoen, and W. Van Dael, J. Chem. Phys. **73**, 4628 (1980).

[45]C. Bagnuls and C. Bervillier, J. Phys. (Paris) Lett. **45**, L95 (1984).

[46]M. E. Fisher, Phys. Rev. Lett. **57**, 1911 (1986).

[47]C. Bagnuls and C. Bervillier (unpublished).

[48]K. E. Newman and E. K. Riedel, Phys. Rev. B **30**, 6615 (1984).

[49]J. Hubbard, Phys. Lett. **39A**, 365 (1972).

[50]A. Aharony and M. E. Fisher, Phys. Rev. B **27**, 4394 (1983).

[51]The incorrectness was pointed out first by Y. Okabe and K. Ideura, Prog. Theor. Phys. **66**, 1959 (1981).

[52]C. Bervillier, Phys. Rev. B **34**, 8141 (1986).